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A Schur transformation for functions in a general class of domains

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To the memory of our teacher, colleague and dear friend Israel Gohberg

Abstract

In this paper we present a framework in which the Schur transformation and the basic interpolation problem for generalized Schur functions, generalized Nevanlinna functions and the like can be studied in a unified way. The basic object is a general class of functions for which a certain kernel has a finite number of negative squares. The results are based on and generalize those in previous papers of the first three authors on the Schur transformation in an indefinite setting.

Keywords: Schur transformation; Generalized Schur function; Generalized Nevanlinna function; Meromorphic function; Projective representation; Pontryagin space; Reproducing kernel; Elementary factor; Basic interpolation problem; Basic boundary interpolation problem; Linear fractional transformation

1. Introduction

1.1

Schur analysis originates from the works of Schur, Herglotz, Tagaki and others (see [30] for reprints of and comments on some of the original papers). It applies, in particular, to the study of

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interpolation problems, moment problems and related questions for classes of complex or matrix valued functions, which are analytic in the open unit disk $\mathbb{D}$ or in an open half-plane and have a constraint on the modulus or on the imaginary or real part. More generally, this constraint is often the assumption that an associated kernel has a finite number of negative squares. Two classes of importance in this context are the generalized Schur functions and the generalized Nevanlinna functions. They were introduced by Krein and Langer, see for instance [32–35], and are defined as follows. A complex function $s(z)$ is a *generalized Schur function* if it is meromorphic in $\mathbb{D}$ and the *Schur kernel*

$$K_{\mathbf{S}}^{s}(z, w) := \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \operatorname{hol}(s) \cap \mathbb{D},$$

has a finite number of negative squares; here and later $\operatorname{hol}(\cdot)$ stands for the common domain of holomorphy of the functions between the brackets. The function $n(z)$ is a *generalized Nevanlinna function* if it is meromorphic in the open upper half-plane $\mathbb{C}_+$ and the *Nevanlinna kernel*

$$K_{\mathbf{N}}^{n}(z, w) := \frac{n(z) - n(w)^*}{z - w^*}, \quad z, w \in \operatorname{hol}(n) \cap \mathbb{C}_+,$$

has a finite number of negative squares.

In these classes of functions a transformation, called the *Schur transformation*, can be defined, and it can be used to solve associated interpolation problems.

In the classical works of Schur [38,39] this is the Carathéodory–Fejér interpolation problem for analytic and contractive functions in $\mathbb{D}$ (these functions bear various names; here we will call them *Schur functions*). The Schur transformation or, more precisely, the inverse Schur transformation, allows to solve these problems in an iterative way by reduction to a one-step problem, which we call the *basic interpolation problem* (see Problem 2.10 below).

1.2

The purpose of this paper is to define a Schur transformation for a general class of functions, that includes the two examples above, and to solve an associated basic interpolation problem. To introduce this general class of functions $\Sigma_\kappa(Q, \rho)$, we follow [13–15]. Let Ω be a connected open subset of the complex plane $\mathbb{C}$ and let $\alpha(z)$, $\beta(z)$ be two complex analytic functions on Ω. We consider the kernel

$$\rho : \Omega \times \Omega \longrightarrow \mathbb{C}, \qquad \rho(z, w) := \alpha(z)\alpha(w)^* - \beta(z)\beta(w)^*, \tag{1.1}$$

assuming that the sets

$$\Omega_+(\rho) := \{z \in \Omega : \rho(z, z) > 0\}, \qquad \Omega_-(\rho) := \{z \in \Omega : \rho(z, z) < 0\}$$

are non-empty. Then, since Ω is connected, also the set

$$\Omega_0(\rho) := \{z \in \Omega : \rho(z, z) = 0\}$$

is non-empty.

In the sequel $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ the set of non-negative integers and $\mathbb{R}$ the set of real numbers. Further, $\mathbb{T}$, $\mathbb{C}^2$ and $\mathbb{C}^{2\times 2}$ stand for the unit circle in $\mathbb{C}$, the set of 2×1 vectors with entries in $\mathbb{C}$ and the set of 2×2 matrices over $\mathbb{C}$, respectively. Finally, I_2 is the identity matrix in $\mathbb{C}^{2\times 2}$ and $J \in \mathbb{C}^{2\times 2}$ is the signature matrix

$$J := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \tag{1.2}$$

here *signature matrix* means that $J = J^* = J^{-1}$.

Definition 1.1. Let $Q(z)$ be a meromorphic $\mathbb{C}^{2\times 2}$-valued function on Ω such that $\det Q(z) \not\equiv 0$, let $\rho(z, w)$ be as in (1.1), and $\kappa \in \mathbb{N}_0$. A complex function $f(z)$ belongs to the class $\Sigma_\kappa(Q, \rho)$ if it is meromorphic in $\Omega_+(\rho)$ and the kernel

$$K^f_{Q,\rho}(z, w) := \frac{\begin{bmatrix} 1 & -f(z) \end{bmatrix} Q(z) J Q(w)^* \begin{bmatrix} 1 & -f(w) \end{bmatrix}^*}{\rho(z, w)},$$
$$z, w \in \operatorname{hol}(f, Q) \cap \Omega_+(\rho), \tag{1.3}$$

has κ negative squares.

For $\kappa = 0$ these classes were studied in [4]. The generalized Schur functions and generalized Nevanlinna functions are special cases of classes $\Sigma_\kappa(Q, \rho)$, as is shown with the following examples.

Example 1.2 (*Generalized Schur Functions*)**.** Let $\Omega = \mathbb{C}$, $\alpha(z) = 1$, $\beta(z) = z$ and $Q(z) = I_2$. Then

$$\rho(z, w) = \rho_{\mathbf{S}}(z, w) := 1 - zw^*,$$

and $\Omega_+(\rho_{\mathbf{S}}) = \mathbb{D}$. According to Definition 1.1, a function $s(z)$ belongs to the class $\Sigma_\kappa(I_2, \rho_{\mathbf{S}})$ if and only if it is meromorphic in $\mathbb{D}$ and the kernel

$$K^s_{I_2,\rho_{\mathbf{S}}}(z, w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \operatorname{hol}(s) \cap \mathbb{D},$$

has κ negative squares. Thus this class $\Sigma_\kappa(I_2, \rho_{\mathbf{S}})$ coincides with the class of generalized Schur functions as introduced above.

For brevity, the following notation will be used:

$$K^s_{\mathbf{S}}(z, w) := K^s_{I_2,\rho_{\mathbf{S}}}(z, w), \qquad \mathbf{S}_\kappa := \Sigma_\kappa(I_2, \rho_{\mathbf{S}}).$$

The class $\mathbf{S}_0$ is the set of all analytic and contractive functions on $\mathbb{D}$; it is called the *Schur class*; the functions in this class are the *Schur functions*.

Example 1.3 (*Generalized Nevanlinna Functions*)**.** Let $\Omega = \mathbb{C}$, $\alpha(z) = 1 - iz$, $\beta(z) = 1 + iz$ and

$$Q = Q_{\mathbf{N}} := \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

Then

$$\rho(z, w) = \rho_{\mathbf{N}}(z, w) := -2i(z - w^*)$$

and $\Omega_+(\rho_{\mathbf{N}}) = \mathbb{C}_+$. According to Definition 1.1, a function $n(z)$ belongs to the class $\Sigma_\kappa(Q_{\mathbf{N}}, \rho_{\mathbf{N}})$ if and only if it is meromorphic in $\mathbb{C}_+$ and the kernel

$$K^n_{Q_{\mathbf{N}},\rho_{\mathbf{N}}}(z, w) = \frac{n(z) - n(w)^*}{z - w^*}, \quad z, w \in \operatorname{hol}(n) \cap \mathbb{C}_+,$$

has κ negative squares. Thus $\Sigma_\kappa(Q_{\mathbf{N}}, \rho_{\mathbf{N}})$ is the class of generalized Nevanlinna functions as introduced above. For brevity, the following notation will be used:

$$K^n_{\mathbf{N}}(z, w) := K^n_{Q_{\mathbf{N}},\rho_{\mathbf{N}}}(z, w), \qquad \mathbf{N}_\kappa := \Sigma_\kappa(Q_{\mathbf{N}}, \rho_{\mathbf{N}}).$$

The class $\mathbf{N}_0$ coincides with the class of all analytic functions on $\mathbb{C}_+$ with the nonnegative imaginary part there. This class is called the *Nevanlinna class* and its elements are *Nevanlinna functions*.

Example 1.4. Consider the kernel

$$\frac{n(z) - n(w)^*}{z - w^*} - \frac{n(z)n(w)^*}{k^2}, \qquad k > 0. \tag{1.4}$$

Meromorphic operator valued functions $n(z)$ on $\mathbb{C}_+$ or $\mathbb{C} \setminus \mathbb{R}$ for which this kernel with $k = 1$ has finitely many negative squares have been introduced and studied in [25,26] in connection with boundary problems with spectral parameter in the boundary conditions. Later, in [18], bi-tangential interpolation problems have been solved in the class of analytic $m\times m$ matrix functions $n(z)$ for which this kernel with fixed $k > 0$ is nonnegative. In the scalar case, which we consider here, the kernel (1.4) is of the form (1.3) with a nonconstant function $Q(z)$:

$$\frac{n(z) - n(w)^*}{z - w^*} - \frac{n(z)n(w)^*}{k^2} = \frac{\begin{bmatrix}1 & -n(z)\end{bmatrix} Q(z) J Q(w)^* \begin{bmatrix}1 & -n(w)\end{bmatrix}^*}{\rho(z, w)},$$

where $\rho(z, w) = \rho_N(z, w) = -2i(z - w^*)$,

$$Q(z) = \frac{1}{k}\begin{bmatrix} k^2 i & k^2 i \\ -1 - iz & 1 - iz\end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

In the scalar case the positivity of the kernel (1.4) in $\mathbb{C} \setminus \mathbb{R}$ means that the function $n(z)$ belongs to the reproducing kernel Hilbert space $\mathcal{L}(n)$ with reproducing kernel $\frac{n(z)-n(w)^*}{z-w^*}$ and has norm less than or equal to k.

The kernel (1.3) belongs to the family of kernels of the form

$$\frac{X(z)\widetilde{J}X(w)^*}{\mathbf{a}(z)J\mathbf{a}(w)^*}, \tag{1.5}$$

where $X(z)$ is a matrix-valued function, $\mathbf{a}(z) = \begin{bmatrix}\alpha(z)\\ \beta(z)\end{bmatrix}$ and $\widetilde{J}$ is also a signature matrix of corresponding size. The study of such kernels was initiated by Lev-Ari [36] and Nudelman [37]. We refer in particular to Lemma 2.1 of the last paper for a characterization of kernels $\rho(z, w)$ of the form (1.1) in terms of factorization properties of associated kernels of the form (1.5) (see also [15, p. 5]). For further details on kernels (1.5) and associated reproducing kernel spaces the reader is referred to, for instance, [15,16,5].

1.3

In this paper we define a Schur transformation and solve the basic interpolation problem for functions in $\Sigma_\kappa(Q, \rho)$ with general $Q(z)$ and $\rho(z, w)$. This is mainly done by a transformation of the class $\Sigma_\kappa(Q, \rho)$ to the class $\mathbf{S}_\kappa$, such that we can use the corresponding results from [2,3,9] and the survey paper [6]; the definition of the Schur transformation in $\mathbf{S}_\kappa$ goes back to Chamfy [22] and Dufresnoy [28].

A short synopsis of the paper is as follows. In Section 2 we introduce the projective version of the class $\Sigma_\kappa(Q,\rho)$. This allows us to work with pairs of holomorphic functions instead of meromorphic functions. We also define the basic interpolation problem and its projective version (see Problem 2.10), which facilitates the possibility of a pole at the given point. In Section 3 we give a review of the Schur transformation and the basic interpolation problem in the class $\mathbf{S}_\kappa$ of generalized Schur functions. These results are taken from the previously mentioned papers [2,3,6]. For the convenience of the reader also some proofs are indicated. The crucial object in this section are polynomial 2×2-matrix functions $\Theta(z)$, which are elementary factors in the class $\mathcal{U}_{\mathbf{S}}$ of all meromorphic $\mathbb{C}^{2\times 2}$-valued functions for which the kernel

$$K_{\mathbf{S}}^{\Theta}(z,w) := \frac{J-\Theta(z)J\Theta(w)^*}{1-zw^*}, \quad z,w\in \mathrm{hol}\,(\Theta)\cap\mathbb{D},$$

has a finite number of negative and positive squares. They generate the fractional linear transformations which solve the basic interpolation problem and whose inverses define the Schur transformation. Section 4 contains a characterization of the classes $\widetilde{\Sigma}_\kappa(Q,\rho)$ and of related classes $\mathcal{U}(Q,\rho)$ (see Definition 2.12) in terms of generalized Schur functions and the corresponding class $\mathcal{U}_{\mathbf{S}}$. This allows us to introduce the Schur transformation and the corresponding basic interpolation problem at an interior point of $\Omega_+(\rho)$ in Section 5. Finally, in Section 6 we discuss the Schur transformation and the basic interpolation problem at a boundary point of $\Omega_+(\rho)$. Since $\Omega_0(\rho)\subset\mathbb{C}$, this discussion does not include the Schur transformation corresponding to the moment problem for Nevanlinna functions; for that we refer the reader to [24,7].

We assume that the reader is familiar with some basic facts on Pontryagin and Krein spaces, reproducing kernel spaces, and kernels with a finite number of positive or negative squares. General references for these notions are, for instance, [1, Chapter 7], [11, Chapter 1] and [27,19,21,31]. The inner product in the Pontryagin spaces in this paper has always a finite number of negative squares; we call this number the *index* of the space. For a function, for example f, of an independent variable z we often write $f(z)$ instead of f.

2. The projective version of $\Sigma_\kappa(Q,\rho)$ and of the basic interpolation problem

2.1. The projective version of the class $\Sigma_\kappa(Q,\rho)$

Since the elements of $\Sigma_\kappa(Q,\rho)$ are meromorphic functions, it is sometimes preferable to deal with a "projective version" of this class. To simplify the notation, we introduce the anti-linear maps $\mathbf{u}\mapsto\mathbf{u}^\times$ on $\mathbb{C}^2$ and $M\mapsto M^\times$ on $\mathbb{C}^{2\times 2}$, defined by

$$\mathbf{u}^\times = \begin{bmatrix} u_1\\ u_2\end{bmatrix}^\times := \begin{bmatrix} u_2^*\\ u_1^*\end{bmatrix}, \qquad M^\times = \begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}^\times = \begin{bmatrix} m_{22}^* & m_{21}^*\\ m_{12}^* & m_{11}^*\end{bmatrix}.$$

Then for $\mathbf{u},\mathbf{v}\in\mathbb{C}^2$ and $M\in\mathbb{C}^{2\times 2}$ we have

$$\mathbf{u}^{\times\times}=\mathbf{u}, \qquad \mathbf{u}^{\times *}\mathbf{v}=\mathbf{v}^{\times *}\mathbf{u}, \qquad (M\mathbf{u})^\times = M^\times\mathbf{u}^\times,$$
$$(M^\times)^\times = M, \qquad (M^*)^\times=(M^\times)^*, \qquad \det(M^\times)=\det(M)^*$$

and with J as in (1.2)

$$M^*JM^\times = M^\times JM^* = \det(M)^*J. \tag{2.1}$$

Definition 2.1. Let $Q(z)$, $\rho(z,w)$, κ be as in Definition 1.1. A $\mathbb{C}^2$-valued function $\mathbf{f}(z)$ is said to belong to the class $\Sigma_\kappa^{\text{proj}}(Q,\rho)$ if it is analytic and non-vanishing in $\Omega_+(\rho)$ and the kernel

$$K^{\mathbf{f}}_{Q,\rho}(z,w) := \frac{\mathbf{f}(z)^{\times *} J Q(z) J Q(w)^* J \mathbf{f}(w)^{\times}}{\rho(z,w)}, \quad z,w \in \operatorname{hol}(Q) \cap \Omega_+(\rho), \tag{2.2}$$

has κ negative squares.

Given a meromorphic function $f(z)$ in $\Omega_+(\rho)$, it can be written as $f(z) = f_1(z)/f_2(z)$, where $f_1(z)$ and $f_2(z)$ are analytic in $\Omega_+(\rho)$ and do not vanish simultaneously. The $\mathbb{C}^2$-valued function

$$\mathbf{f}(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$$

will be referred to as a *projective representation* of $f(z)$. Note that between the kernels $K^{\mathbf{f}}_{Q,\rho}(z,w)$ and $K^{f}_{Q,\rho}(z,w)$, defined in (1.3) and (2.2), respectively, the following relation holds:

$$\begin{aligned} K^{\mathbf{f}}_{Q,\rho}(z,w) &= \frac{\begin{bmatrix} f_2(z) & -f_1(z) \end{bmatrix} Q(z) J Q(w)^* \begin{bmatrix} f_2(w) & -f_1(w) \end{bmatrix}^*}{\rho(z,w)} \\ &= f_2(z) K^{f}_{Q,\rho}(z,w) f_2(w)^*, \quad z,w \in \operatorname{hol}(Q) \cap \Omega_+(\rho). \end{aligned}$$

Hence the meromorphic function $f(z)$ belongs to the class $\Sigma_\kappa(Q,\rho)$ if and only if some (or any) projective representation $\mathbf{f}(z)$ of $f(z)$ belongs to the class $\Sigma_\kappa^{\text{proj}}(Q,\rho)$. We mention that the class $\Sigma_\kappa^{\text{proj}}(Q,\rho)$ may contain functions which are *not* projective representations of any meromorphic function with values in $\mathbb{C}$; see Definition 2.4 and Example 2.6 below.

Example 2.2. Definition 2.1 implies that a $\mathbb{C}^2$-valued function

$$\mathbf{f}(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$$

belongs to the class $\Sigma_\kappa^{\text{proj}}(I_2,\rho)$ if and only if it is analytic and non-vanishing in $\Omega_+(\rho)$ and the kernel

$$K^{\mathbf{f}}_{I_2,\rho}(z,w) = \frac{f_2(z) f_2(w)^* - f_1(z) f_1(w)^*}{\rho(z,w)}$$

has κ negative squares in $\Omega_+(\rho)$. In this case the zero set of the function $f_2(z)$ cannot have accumulation points in $\Omega_+(\rho)$ (otherwise the kernel $K^{\mathbf{f}}_{I_2,\rho}(z,w)$ would have infinitely many negative squares in $\Omega_+(\rho)$), and, therefore, $\mathbf{f}(z)$ is a projective representation of the meromorphic function $f(z) := f_1(z)/f_2(z)$ from the class $\Sigma_\kappa(I_2,\rho)$. In particular, a $\mathbb{C}^2$-valued function

$$\mathbf{s}(z) = \begin{bmatrix} s_1(z) \\ s_2(z) \end{bmatrix}$$

belongs to the projective Schur class

$$\mathbf{S}_\kappa^{\text{proj}} := \Sigma_\kappa^{\text{proj}}(I_2, \rho_{\mathbf{S}})$$

if and only if it is analytic and non-vanishing in $\mathbb{D}$ and $\mathbf{s}(z)$ is a projective representation of the meromorphic function $s(z) := s_1(z)/s_2(z)$ from the Schur class $\mathbf{S}_\kappa$. In the sequel the kernel associated with the vector function $\mathbf{s}(z)$ will be denoted by $K^{\mathbf{s}}_{\mathbf{S}}(z,w)$. It is given by

$$K^{\mathbf{s}}_{\mathbf{S}}(z,w) := \frac{s_2(z)s_2(w)^* - s_1(z)s_1(w)^*}{1 - zw^*}, \quad z,w \in \mathbb{D}.$$

Example 2.3. Denote by $\mathbf{N}_\kappa^{\mathrm{proj}}$ the projective Nevanlinna class:

$$\mathbf{N}_\kappa^{\mathrm{proj}} := \Sigma_\kappa^{\mathrm{proj}}(Q_{\mathbf{N}}, \rho_{\mathbf{N}}).$$

According to Definition 2.1, a $\mathbb{C}^2$-valued function

$$\mathbf{n}(z) = \begin{bmatrix} n_1(z) \\ n_2(z) \end{bmatrix}$$

belongs to $\mathbf{N}_\kappa^{\mathrm{proj}}$ if and only if it is analytic and non-vanishing in $\mathbb{C}_+$ and the kernel

$$K_{\mathbf{N}}^{\mathbf{n}}(z, w) := \frac{n_1(z)n_2(w)^* - n_2(z)n_1(w)^*}{z - w^*}$$

has κ negative squares in $\mathbb{C}_+$. In this case either

(1) $n_2 \equiv 0$, n_1 is an arbitrary function, analytic and non-vanishing in $\mathbb{C}_+$, and $\kappa = 0$, or
(2) $\mathbf{n}(z)$ is a projective representation of the meromorphic function $n(z) := n_1(z)/n_2(z)$ from the Nevanlinna class $\mathbf{N}_\kappa$.

From the projective point of view it is natural to extend the definition of the class $\Sigma_\kappa(Q, \rho)$ as follows.

Definition 2.4. Let

$$\mathbf{f}(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$$

be a $\mathbb{C}^2$-valued function. Then the function

$$f(z) = \begin{cases} f_1(z)/f_2(z), & f_2(z) \neq 0, \\ \infty, & f_2(z) = 0, \end{cases}$$

is said to belong to the class $\widetilde{\Sigma}_\kappa(Q, \rho)$ if $\mathbf{f}(z)$ belongs to $\Sigma_\kappa^{\mathrm{proj}}(Q, \rho)$. In this case $\mathbf{f}(z)$ is called a projective representation of $f(z)$.

Example 2.5. The class $\Sigma_\kappa(I_2, \rho)$ and its extension $\widetilde{\Sigma}_\kappa(I_2, \rho)$ coincide (see Example 2.2). In particular, the extended Schur class $\widetilde{\mathbf{S}}_\kappa := \widetilde{\Sigma}_\kappa(I_2, \rho_{\mathbf{S}})$ coincides with the class $\mathbf{S}_\kappa$.

Example 2.6. Denote by $\widetilde{\mathbf{N}}_\kappa$ the extended Nevanlinna class:

$$\widetilde{\mathbf{N}}_\kappa := \widetilde{\Sigma}_\kappa(Q_{\mathbf{N}}, \rho_{\mathbf{N}}).$$

Then, in view of Example 2.3,

$$\widetilde{\mathbf{N}}_\kappa = \begin{cases} \mathbf{N}_\kappa & \text{if } \kappa \geq 1, \\ \mathbf{N}_0 \cup \{\infty\} & \text{if } \kappa = 0; \end{cases}$$

here ∞ stands for the function $n(z) \equiv \infty$. It has a projective representation

$$\mathbf{n}(z) \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which belongs to the projective Nevanlinna class $\mathbf{N}_0^{\mathrm{proj}}$.

Example 2.7. Consider the kernel

$$\rho_1 : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}, \qquad \rho_1(z, w) = z^2 + w^{*2} - z^2 w^{*2}.$$

Since

$$z^2 + w^{*2} - z^2 w^{*2} = 1 - (1 - z^2)(1 - w^2)^*,$$

the kernel $\rho_1(z, w)$ is of the form (1.1). Furthermore,

$$\Omega_+(\rho_1) = \left\{ z : |1 - z^2| < 1 \right\} = \Omega_r \cup \Omega_l,$$

where Ω_r and Ω_l are the images of the open disk $\{z : |z - 1| < 1\}$ under the maps $z \mapsto \sqrt{z}$ and $z \mapsto -\sqrt{z}$, respectively. Here and below $\sqrt{z}$ denotes the principal branch of the square root. Note that $\Omega_r \cap \Omega_l = \emptyset$ and that

$$\sqrt{z^2} = \begin{cases} z, & z \in \Omega_r, \\ -z, & z \in \Omega_l. \end{cases}$$

The function

$$f(z) := \begin{cases} \infty, & z \in \Omega_r, \\ 0, & z \in \Omega_l, \end{cases}$$

belongs to the class $\widetilde{\Sigma}_0(Q_1, \rho_1)$, where

$$Q_1(z) := \begin{bmatrix} 1 & \frac{z}{2} \\ 1 & -\frac{z}{2} \end{bmatrix}.$$

Indeed, the function

$$\mathbf{f}(z) := \begin{cases} \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & z \in \Omega_r, \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & z \in \Omega_l, \end{cases}$$

which is a projective representation of $f(z)$, satisfies the relation

$$\mathbf{f}(z)^{\times *} J Q_1(z) = \begin{bmatrix} 1 & -\frac{\sqrt{z^2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & -s(1 - z^2) \end{bmatrix}, \quad z \in \Omega_+(\rho_1),$$

where the function

$$s(z) = \frac{\sqrt{1 - z}}{2}, \quad z \in \mathbb{D},$$

is analytic and contractive in $\mathbb{D}$. Hence $s(z) \in \mathbf{S}_0$ and the kernel $K^s_{\mathbf{S}}(z, w)$ is positive in $\mathbb{D}$. Since

$$K^{\mathbf{f}}_{Q_1,\rho_1}(z, w) = \frac{1 - s(1 - z^2)s(1 - w^2)^*}{1 - (1 - z^2)(1 - w^2)^*} = K^s_{\mathbf{S}}(1 - z^2, 1 - w^2), \quad z, w \in \Omega_+(\rho_1),$$

the kernel $K^{\mathbf{f}}_{Q_1,\rho_1}(z, w)$ is positive in $\Omega_+(\rho_1)$. Therefore $\mathbf{f} \in \Sigma_0^{\mathrm{proj}}(Q_1, \rho_1)$ and, according to Definition 2.4, $f \in \widetilde{\Sigma}_0(Q_1, \rho_1)$.

2.2. The basic interpolation problem

Recall that the basic interpolation problem for a class $\mathbf{S}_\kappa$ of generalized Schur functions in the simplest case is as follows. *Given $z_1 \in \mathbb{D}$ and a number $c_0 \in \mathbb{C}$, find all functions $s(z) \in \mathbf{S}_\kappa$ which are analytic at z_1 and such that $s(z_1) = c_0$.* If $\kappa = 0$, this problem has infinitely many solutions if $|c_0| < 1$, exactly one solution if $|c_0| = 1$, and no solution if $|c_0| > 1$. In the first case the solutions can be described by a fractional linear transformation with the elements of the Schur class $\mathbf{S}_0$ as parameters. In the context of repeated Schur transformations the number c_0 is the so-called *first Schur parameter*.

If $\kappa \geq 1$ and $|c_0| \neq 1$ there are infinitely many solutions $s(z)$ of the problem, which can also be described by a fractional linear transformation with a certain Schur class as parameter set; see Theorem 3.4 below. If $\kappa \geq 1$ and $|c_0| = 1$, the situation becomes more intriguing. In this case we can choose an arbitrary number $c_1 \in \mathbb{C}, c_1 \neq 0$, and there exist infinitely many functions $s(z) \in \mathbf{S}_\kappa$ which are analytic at z_1 and such that

$$s(z_1) = c_0, \qquad s'(z_1) = c_1;$$

they are again given through a fractional linear transformation with, roughly, the set $\mathbf{S}_{\kappa-1}$ as a parameter set. If $c_1 = 0$, or, more generally,

$$c_1 = c_2 = \cdots = c_{k-1} = 0 \quad \text{for some } k \in \mathbb{N},$$

we can choose numbers $c_k \neq 0, c_{k+1}, \ldots, c_{2k-1} \in \mathbb{C}$, and for each $\kappa \geq k$ there are infinitely many functions $s(z) \in \mathbf{S}_\kappa$ such that

$$s(z) = c_0 + c_k(z - z_1)^k + \cdots + c_{2k-1}(z - z_1)^{2k-1} + \mathrm{O}((z - z_1)^{2k}), \tag{2.3}$$

again described by a fractional linear transformation with, roughly, the elements of $\mathbf{S}_{\kappa-k}$ as parameters. Observe that for the solution $s(z)$ in (2.3) the function $s(z) - c_0$ has at z_1 a zero of order k. We call the numbers $c_0, c_k, c_{k+1}, \ldots, c_{2k-1}$ the *first augmented Schur parameter*. It should be mentioned, that this extension of the interpolation problem is natural in the sense that the transformation matrices which describe the solutions of the problem are elementary factors in the class $\mathcal{U}_{\mathbf{S}}$; see Section 2.3.

For the following, if $|c_0| = 1$, with the parameters $c_k, c_{k+1}, \ldots, c_{2k-1},\ c_k \neq 0$, we introduce the polynomial

$$q(z) := \frac{1}{c_0}\left(c_k + c_{k+1}(z - z_1) + \cdots + c_{2k-1}(z - z_1)^{k-1}\right);$$

it has the properties $q(z_1) \neq 0$ and $\deg(q) < k$. Then the relation (2.3) becomes

$$s(z) = c_0 + c_0 q(z)(z - z_1)^k + \mathrm{O}((z - z_1)^{2k}), \quad z \to 0.$$

In projective terms, in view of the fact that the classes $\mathbf{S}_\kappa$ and $\widetilde{\mathbf{S}}_\kappa$ coincide (see Examples 2.2 and 2.5), the basic interpolation problem in $\widetilde{\mathbf{S}}_\kappa$ at $z_1 \in \mathbb{D}$ in its simplest form can be formulated as follows. For given $c_0 \in \mathbb{C} \cup \{\infty\}$ introduce the vector

$$\mathbf{u}_0 := \begin{cases} \begin{bmatrix} 1 & c_0 \end{bmatrix}^*, & c_0 \in \mathbb{C}, \\ \begin{bmatrix} 0 & 1 \end{bmatrix}^*, & c_0 = \infty. \end{cases} \tag{2.4}$$

Problem 2.8. Find all $\mathbf{s}(z) \in \tilde{\mathbf{S}}_\kappa$ such that

$$\mathbf{u}_0^* J \mathbf{s}(z_1) = 0. \tag{2.5}$$

With the two choices of $\mathbf{u}_0$ in (2.4) and $\mathbf{s}(z) = \begin{bmatrix} s_1(z) \\ s_2(z) \end{bmatrix}$ the condition (2.5) takes the form

$$s_1(z_1) = c_0 s_2(z_1) \quad (\text{if } c_0 \in \mathbb{C}) \quad \text{or} \quad s_2(z_1) = 0 \quad (\text{if } c_0 = \infty)$$

or, equivalently,

$$s(z) \text{ is analytic at } z = 0 \quad \text{and} \quad s(0) = c_0 \quad \text{or} \quad s(z) \text{ has a pole at } z = 0.$$

The alternative $|c_0| \neq 1$ or $|c_0| = 1$ means now $\mathbf{u}_0^* J \mathbf{u}_0 \neq 0$ or $\mathbf{u}_0^* J \mathbf{u}_0 = 0$, that is, the description of the solutions of Problem 2.8 in terms of a linear fractional transformation depends on whether $\mathbf{u}_0$ is J-neutral or not. In the case when $\mathbf{u}_0$ is not J-neutral, we describe all solutions of Problem 2.8 in Theorem 3.4 below; if $\mathbf{u}_0$ is J-neutral, the only solution of the problem in the case $\kappa = 0$ is the function $s(z) \equiv c_0$, in the case $\kappa > 0$ non-trivial solutions exist and they are described for a given augmented Schur parameter by a linear fractional transformation. This augmented Schur parameter consists of a J-neutral vector $\mathbf{u}_0$, $k \in \mathbb{N}$, and a polynomial $q(z)$ of degree $< k$ with $q(z_1) \neq 0$. With the matrix

$$P_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(I_2 - J), \tag{2.6}$$

the interpolation problem is formulated as follows.

Problem 2.9. Find all functions $s(z) \in \mathbf{S}_\kappa$ such that any projective representation $\mathbf{s}(z)$ of $s(z)$ satisfies

$$\frac{\mathbf{u}_0^* J \mathbf{s}(z)}{\mathbf{u}_0^* P_2 \mathbf{s}(z)} = q(z)(z - z_1)^k + \mathrm{O}((z - z_1)^{2k}), \quad z \to 0.$$

The solutions of this problem will be given in Theorem 3.6.

Finally we formulate the basic interpolation problem for the case of a general domain. Here we restrict ourselves to the case $|c_0| \neq 1$. The formulation of the problem in the case of an augmented Schur parameter needs as preparation a reduction to the case of the unit disk which will be given only in Section 4. In fact, the definition of this class of solutions would coincide with its description in (1) of Theorem 5.5. In the following definitions, $Q(z)$ and $\rho(z, w)$ are as in Definition 1.1.

Problem 2.10. Let $z_1 \in \operatorname{hol}(Q, Q^{-1}) \cap \Omega_+(\rho)$ and $c_0 \in \mathbb{C} \cup \{\infty\}$, $|c_0| \neq 1$ be given. Find all $f(z) \in \widetilde{\Sigma}_\kappa(Q, \rho)$ such that $f(z_1) = c_0$.

Problem 2.10 can also be formulated in projective terms. To this end, given $c_0 \in \mathbb{C} \cup \{\infty\}$ we introduce $\mathbf{u}_0$ as in (2.4). Then $f(z) \in \widetilde{\Sigma}_\kappa(Q, \rho)$ satisfies the interpolation condition $f(z_1) = c_0$ if and only if any projective representation $\mathbf{f}(z) \in \Sigma_\kappa^{\mathrm{proj}}(Q, \rho)$ of $f(z)$ satisfies

$$\mathbf{u}_0^* J \mathbf{f}(z_1) = 0. \tag{2.7}$$

This leads to the following projective version of Problem 2.10.

Problem 2.11. Let $z_1 \in \operatorname{hol}(Q, Q^{-1}) \cap \Omega_+(\rho)$ and $\mathbf{u}_0 \in \mathbb{C}^2 \setminus \{0\}$, $\mathbf{u}_0^* J \mathbf{u}_0 \neq 0$, be given. Find all $f(z) \in \widetilde{\Sigma}_\kappa(Q, \rho)$ such that any projective representation $\mathbf{f}(z)$ of $f(z)$ satisfies (2.7).

2.3. The class $\mathcal{U}(Q, \rho)$

The solutions of Problem 2.11 will be presented in terms of linear fractional transformations. We define the linear fractional transformation T_M of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ associated with a non-degenerate 2×2 complex matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

by the formula

$$T_M(z) := \frac{m_{11}z + m_{12}}{m_{21}z + m_{22}}.$$

Note that

$$T_{cM} = T_M, \quad c \in \mathbb{C} \setminus \{0\},$$

and that linear fractional transformations enjoy the multiplicative property

$$T_{M_1} T_{M_2} = T_{M_1 M_2},$$

which implies, in particular, that if M is invertible, then T_M is invertible and

$$T_M^{-1} = T_{M^{-1}}.$$

A crucial role in the sequel is played by the following class of matrix-valued functions.

Definition 2.12. The $\mathbb{C}^{2\times 2}$-valued function $\Theta(z)$ belongs to the class $\mathcal{U}(Q, \rho)$ if it is meromorphic in $\Omega_+(\rho)$ and the kernel

$$K^{\Theta}_{Q,\rho}(z, w) := \frac{Q(z)JQ(w)^* - \Theta(z)Q(z)JQ(w)^*\Theta(w)^*}{\rho(z, w)},$$
$$z, w \in \operatorname{hol}(\Theta, Q) \cap \Omega_+(\rho), \tag{2.8}$$

has finitely many positive and negative squares.

Definition 2.12 can also be interpreted as follows. A function $\Theta(z)$ belongs to the class $\mathcal{U}(Q, \rho)$ if and only if it is meromorphic in $\Omega_+(\rho)$ and (2.8) is the reproducing kernel of a finite-dimensional Pontryagin space, the dimension being the sum of the numbers of positive and negative squares of the kernel. In the sequel, if $K(z, w)$ is a kernel with finitely many negative squares, $\mathcal{P}(K)$ denotes the Pontryagin space with the reproducing kernel $K(z, w)$.

According to Definition 2.12, a $\mathbb{C}^{2\times 2}$-valued function $\Theta(z)$ belongs to the class $\mathcal{U}_{\mathbf{S}} := \mathcal{U}(I_2, \rho_{\mathbf{S}})$ if it is meromorphic in $\mathbb{D}$ and the kernel

$$K^{\Theta}_{\mathbf{S}}(z, w) := \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}, \quad z, w \in \operatorname{hol}(\Theta) \cap \mathbb{D},$$

has finitely many positive and negative squares. According to [17, Theorem 8.1], this is the case if and only if $\Theta(z)$ is a rational matrix function, which is J-unitary on $\mathbb{T}$:

$$\Theta(t)J\Theta(t)^* = J, \quad t \in \operatorname{hol}(\Theta) \cap \mathbb{T}, \tag{2.9}$$

or, what is the same: for all $z \in \mathbb{C} \setminus \{0\}$ with $z, 1/z^* \in \operatorname{hol}(\Theta)$ we have

$$\Theta(z)J\Theta(1/z^*)^* = J.$$

Note that a rational matrix function $\Theta(z)$, which satisfies (2.9), may have poles on $\mathbb{T}$. For example, the rational matrix function

$$\Theta(z) = \begin{bmatrix} \dfrac{2z}{z-1} & \dfrac{1+z}{1-z} \\ \dfrac{z+1}{z-1} & \dfrac{2}{1-z} \end{bmatrix}$$

is J-unitary on the unit circle for $z \neq 1$ (and belongs to the class $\mathcal{U}_{\mathbf{S}}$).

Example 2.13. Set

$$J_{\mathbf{N}} := \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -\frac{1}{2} Q_{\mathbf{N}} J Q_{\mathbf{N}}^*.$$

According to Definition 2.12, a $\mathbb{C}^{2\times 2}$-valued function $\Theta(z)$ belongs to the class $\mathcal{U}_{\mathbf{N}} := \mathcal{U}(Q_{\mathbf{N}}, \rho_{\mathbf{N}})$ if it is meromorphic in $\mathbb{C}_+$ and the kernel

$$K_{\mathbf{N}}^{\Theta}(z, w) := \frac{J_{\mathbf{N}} - \Theta(z) J_{\mathbf{N}} \Theta(w)^*}{i(z - w^*)}, \quad z, w \in \operatorname{hol}(\Theta) \cap \mathbb{C}_+,$$

has finitely many positive and negative squares. As for the circle case, and according to [17, Theorem 8.1], this is the case if and only if $\Theta(z)$ is a rational matrix function, which is $J_{\mathbf{N}}$-unitary on $\mathbb{R}$:

$$\Theta(x) J_{\mathbf{N}} \Theta(x)^* = J_{\mathbf{N}}, \quad x \in \operatorname{hol}(\Theta) \cap \mathbb{R},$$

or, equivalently,

$$\Theta(z) J_{\mathbf{N}} \Theta(z^*)^* = J_{\mathbf{N}}, \quad z, z^* \in \operatorname{hol} \Theta.$$

Note that a rational matrix function $\Theta(z)$, which is $J_{\mathbf{N}}$-unitary on $\mathbb{R} \cap \operatorname{hol} \Theta$, may have poles on $\mathbb{R}$. An example is the rational matrix function

$$\Theta(z) = \begin{bmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{bmatrix},$$

which belongs to the class $\mathcal{U}_{\mathbf{N}}$.

Definition 2.14. The function $\Theta(z) \in \mathcal{U}(Q, \rho)$ is an elementary factor in $\mathcal{U}(Q, \rho)$ if the space $\mathcal{P}(K_{Q,\rho}^{\Theta})$ contains no non-trivial subspace of the form $\mathcal{P}(K_{Q,\rho}^{\Theta_1})$ with some $\Theta_1(z) \in \mathcal{U}(Q, \rho)$.

In Theorems 5.4 and 5.5 below we show that all solutions of Problem 2.11 are of the form

$$f(z) = T_{\Theta(z)}(\hat{f}(z)), \tag{2.10}$$

where $\Theta(z)$ is an elementary factor in the class $\mathcal{U}(Q, \rho)$ and $\hat{f}(z)$ is a function from the class $\widetilde{\Sigma}_{\hat{\kappa}}(Q, \rho)$ with $\hat{\kappa} \leq \kappa$. The inverse transformation

$$\hat{f}(z) = T_{\Phi(z)}(f(z)), \tag{2.11}$$

where $\Phi(z) = \Theta(z)^{-1}$, is the Schur transformation (centered at z_1) in the class $\widetilde{\Sigma}_{\kappa}(Q, \rho)$. The identities (2.10) and (2.11) should be understood in terms of projective representations: if

$$\mathbf{f}(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{f}}(z) = \begin{bmatrix} \hat{f}_1(z) \\ \hat{f}_2(z) \end{bmatrix}$$

are projective representations of $f(z)$ and $\hat{f}(z)$, respectively, then for example (2.10) means that

$$f_1(z)\big(\theta_{21}(z)\hat{f}_1(z)+\theta_{22}(z)\hat{f}_2(z)\big)=f_2(z)\big(\theta_{11}(z)\hat{f}_1(z)+\theta_{12}(z)\hat{f}_2(z)\big),$$

where

$$\Theta(z)=\begin{bmatrix}\theta_{11}(z) & \theta_{12}(z)\\ \theta_{21}(z) & \theta_{22}(z)\end{bmatrix}.$$

3. The Schur transformation and the basic interpolation problem in the class of generalized Schur functions

In this section, we collect some results from the papers [2,3,6] on the basic interpolation problem and elementary factors.

3.1. J-unitary polynomials

The definition of the Schur transformation centered at $z_1 = 0$ in the class $\mathbf{S}_\kappa$ is based on polynomials in $\mathcal{U}_\mathbf{S}$. First we introduce some notation. Given a rational matrix function $\Theta(z)$, denote by $\deg(\Theta)$ the *McMillan degree* of $\Theta(z)$ and by $\Theta^\#(z)$ the matrix function

$$\Theta^\#(z):=\Theta(1/z^*)^*,\quad 1/z^*\in \operatorname{hol}\Theta.$$

Furthermore, with J as in (1.2) denote by $\mathcal{H}^2_J$ the vector space of square-summable power series

$$\mathbf{h}(z)=\sum_{j=0}^{\infty}\mathbf{h}_j z^j,\quad \text{where } \mathbf{h}_0,\mathbf{h}_1,\mathbf{h}_2,\ldots\in\mathbb{C}^2 \text{ are such that } \sum_{j=0}^{\infty}\mathbf{h}_j^*\mathbf{h}_j<\infty,$$

equipped with the indefinite inner product

$$\left\langle \sum_{j=0}^{\infty}\mathbf{h}_j z^j,\sum_{\ell=0}^{\infty}\mathbf{g}_\ell z^\ell\right\rangle_{\mathcal{H}^2_J}:=\sum_{k=0}^{\infty}\mathbf{g}_k^* J\mathbf{h}_k.$$

Note that the space $\mathcal{H}^2_J$ is a reproducing kernel Krein space with the $\mathbb{C}^{2\times 2}$-valued reproducing kernel $J/(1-zw^*)$.

Theorem 3.1. *Let $\Theta(z)$ be a $\mathbb{C}^{2\times 2}$-valued polynomial. Then $\Theta(z)$ belongs to the class $\mathcal{U}_\mathbf{S}$ if and only if $\Theta(z)$ is J-unitary on $\mathbb{T}$. In this case the following statements hold.*

(1) $\mathcal{P}(K_\mathbf{S}^\Theta)=\mathcal{H}^2_J\ominus\Theta\mathcal{H}^2_J$. *In particular, the space $\mathcal{P}(K_\mathbf{S}^\Theta)$ is isometrically included in $\mathcal{H}^2_J$.*

(2) $\dim\mathcal{P}(K_\mathbf{S}^\Theta)=\deg(\Theta)$.

(3) *The elements of $\mathcal{P}(K_\mathbf{S}^\Theta)$ are $\mathbb{C}^2$-valued polynomials.*

(4) *A $\mathbb{C}^2$-valued polynomial $\mathbf{h}(z)$ belongs to $\mathcal{P}(K_\mathbf{S}^\Theta)$ if and only if*

$$\lim_{z\to 0}\mathbf{h}^\#(z)J\Theta(z)=0.$$

(5) *The space $\mathcal{P}(K_\mathbf{S}^\Theta)$ is invariant for the backward shift operator R_0 defined by*

$$(R_0\mathbf{f})(z):=\frac{\mathbf{f}(z)-\mathbf{f}(0)}{z},\quad \mathbf{f}\in\mathcal{P}(K_\mathbf{S}^\Theta).$$

Proof. The statement that $\Theta(z)$ belongs to the class $\mathcal{U}_\mathbf{S}$ if and only if (2.9) holds follows, for instance, from [17, Theorem 8.1]. Statements (1), (2), (3) and (5) can be found in

[2, Theorem 4.2]. Statement (4) is proved as follows. In view of statement (1) and the fact that $\mathbb{C}^2$-valued polynomials form a dense linear subset of $\mathcal{H}^2_J$, a $\mathbb{C}^2$-valued polynomial $\mathbf{h}(z)$ belongs to $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ if and only if

$$\langle \Theta\mathbf{f}, \mathbf{h}\rangle_{\mathcal{H}^2_J} = 0$$

for every function $\mathbf{f}(z)$ of the form

$$\mathbf{f}(z) = \mathbf{v}z^k, \quad \mathbf{v} \in \mathbb{C}^2,\ k \in \mathbb{N}_0. \tag{3.1}$$

Write

$$\mathbf{h}(z) = \sum_{j=0}^{\infty} \mathbf{h}_j z^j, \qquad \Theta(z) = \sum_{j=0}^{\infty} \Theta_j z^j$$

(in these expansions all but finitely many terms are zeros) and observe that for a function $\mathbf{f}(z)$ of the form (3.1) one has

$$\langle \Theta\mathbf{f}, \mathbf{h}\rangle_{\mathcal{H}^2_J} = \sum_{j=0}^{\infty} \mathbf{h}^*_{j+k} J\Theta_j \mathbf{v},$$

which is exactly the coefficient of z^{-k} in the Laurent expansion of $\mathbf{h}^{\#}(z)J\Theta(z)\mathbf{v}$ at $z = 0$. Thus $\mathbf{h}(z)$ belongs to $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ if and only if the rational matrix function $\mathbf{h}^{\#}(z)J\Theta(z)$ is analytic at $z = 0$ and vanishes there. $\square$

The following theorem can be found in [2, Proposition 5.1]. Recall that a vector $\mathbf{u} \in \mathbb{C}^2$ is J-neutral if $\mathbf{u}^*J\mathbf{u} = 0$; the matrix P_2 was defined in (2.6).

Theorem 3.2. *Every non-constant polynomial elementary factor $\Theta(z)$ in the class $\mathcal{U}_{\mathbf{S}}$ is, up to multiplication by a J-unitary constant matrix on the right, of one of the following two forms:*

(1)

$$\Theta(z) = I_2 - (1 - z)\frac{\mathbf{u}\mathbf{u}^*J}{\mathbf{u}^*J\mathbf{u}}, \tag{3.2}$$

where $\mathbf{u} \in \mathbb{C}^2$ is not J-neutral. In this case the associated reproducing kernel Pontryagin space $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ is one-dimensional:

$$\mathcal{P}(K^{\Theta}_{\mathbf{S}}) = \operatorname{span}_{\mathbb{C}}\{\mathbf{u}\}.$$

(2)

$$\Theta(z) = z^k I_2 + (p(z) - z^{2k}p^{\#}(z))\frac{\mathbf{u}\mathbf{u}^*J}{\mathbf{u}^*P_2\mathbf{u}}, \tag{3.3}$$

where $\mathbf{u} \in \mathbb{C}^2$ is non-zero and J-neutral, $k \in \mathbb{N}_0$, and $p(z)$ is a polynomial with $p(0) \neq 0$ and $\deg(p) < k$. In this case the kernel $K^{\Theta}_{\mathbf{S}}(z, w)$ has k positive and k negative squares in $\mathbb{D}$. The associated reproducing kernel Pontryagin space $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ can be characterized as follows:

$$\mathcal{P}(K^{\Theta}_{\mathbf{S}}) = \operatorname{span}_{\mathbb{C}}\{\mathbf{h}(z), R_0\mathbf{h}(z), \ldots, R_0^{2k-1}\mathbf{h}(z)\},$$

where $\mathbf{h}(z)$ is the $\mathbb{C}^2$-valued polynomial

$$\mathbf{h}(z) := (z^{2k-1}I_2 + z^{k-1}q^{\#}(z)P_2)\mathbf{u}, \tag{3.4}$$

and $q(z)$ *is the polynomial determined by*

$$\deg(q) < k \quad \textit{and} \quad p(z)q(z) = 1 + \mathrm{O}(z^k), \quad z \to 0.$$

To prove the theorem we use the following result, which is the finite dimensional version of results of Ball and Helton [20, Theorem 2.1], de Branges [23, Theorem III], and Dym [29, Theorems 4.1 and 4.2]. It is obtained as a special case of [17, Theorem 6.12] by using [17, Theorem 6.11] in the setting of polynomials.

Theorem 3.3. *Every non-trivial, non-degenerate,* R_0*-invariant and finite dimensional subspace* $\mathcal{P}$ *of* $\mathcal{H}_J^2$*, whose elements are polynomials, is of the form* $\mathcal{P} = \mathcal{P}(K_{\mathbf{S}}^{\Theta})$ *for some polynomial* $\Theta(z)$ *from the class* $\mathcal{U}_{\mathbf{S}}$*.*

Proof of Theorem 3.2. Let $\Theta(z)$ be a non-constant polynomial elementary factor from the class $\mathcal{U}_{\mathbf{S}}$. Then, according to statement (5) of Theorem 3.1, the finite-dimensional space $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ is R_0-invariant and hence contains an eigenvector $\mathbf{u}(z)$ of the operator R_0. Since, by statement (3) of Theorem 3.1, $\mathbf{u}(z)$ is a polynomial, the eigenvalue is 0 and $\mathbf{u}(z)$ is constant:

$$\mathbf{u}(z) \equiv \mathbf{u} \in \mathbb{C}^2 \setminus \{0\}.$$

First assume that the vector $\mathbf{u}$ is not J-neutral. Then the one-dimensional $\mathrm{span}_{\mathbb{C}}\{\mathbf{u}\}$ is a non-degenerate subspace of $\mathcal{H}_J^2$ and, in view of Theorem 3.3,

$$\mathcal{P}(K_{\mathbf{S}}^{\Theta}) = \mathrm{span}_{\mathbb{C}}\{\mathbf{u}\}.$$

In this case the constant kernel

$$\frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^* J \mathbf{u}}$$

is the reproducing kernel of the space $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ and thus

$$\frac{J - \Theta(z) J \Theta(w)^*}{1 - z w^*} = \frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^* J \mathbf{u}}. \tag{3.5}$$

In view of identity (2.9),

$$\Theta(1)^* J \Theta(1) = J$$

and therefore, setting $w = 1$ in (3.5) leads to

$$\Theta(z) = \left(I_2 - (1 - z) \frac{\mathbf{u}\mathbf{u}^* J}{\mathbf{u}^* J \mathbf{u}} \right) \Theta(1).$$

Thus $\Theta(z)$ is, up to multiplication from the right by a J-unitary constant matrix, of the form (3.2).

Now assume that the nonzero vector $\mathbf{u}$ is J-neutral. Then the space $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ contains no constant non-J-neutral vectors (otherwise, as was already shown, $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ is one-dimensional, which leads to a contradiction). Therefore, on account of Theorem 3.3, $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ is spanned by a single chain:

$$\mathcal{P}(K_{\mathbf{S}}^{\Theta}) = \mathrm{span}_{\mathbb{C}}\{\mathbf{h}_0(z), \ldots, \mathbf{h}_m(z)\},$$

where $m \in \mathbb{N}_0$ and $\mathbf{h}_0(z), \ldots, \mathbf{h}_m(z)$ are the $\mathbb{C}^2$-valued polynomials

$$\mathbf{h}_0(z) = \mathbf{u}, \qquad \mathbf{h}_{j-1}(z) = R_0 \mathbf{h}_j(z), \quad j = 1, \ldots, m.$$

Then the polynomials $\mathbf{h}_j(z)$ are linearly independent and satisfy the relations

$$\mathbf{h}_j(z) = z\mathbf{h}_{j-1}(z) + \mathbf{h}_j(0), \quad j = 1, \dots, m.$$

Moreover, one can always modify the polynomials $\mathbf{h}_1(z), \dots, \mathbf{h}_m(z)$ by adding suitable multiples of $\mathbf{u}$, so that the first component of $\mathbf{h}_j(0)$ is canceled and

$$\mathbf{h}_j(z) = z\mathbf{h}_{j-1}(z) + c_j^* P_2\mathbf{u}, \quad j = 1, \dots, m, \tag{3.6}$$

for some complex numbers $c_1, \dots, c_m$. This is possible since both entries of $\mathbf{u}$ are nonzero. Let k be the smallest positive integer such that $c_k \neq 0$. Then

$$\langle \mathbf{h}_j, \mathbf{h}_\ell \rangle_{\mathcal{H}_J^2} = 0, \quad 0 \le j, \ell \le k-1,$$

and the fact that $\mathcal{P}(K_{\mathbf{S}}^\Theta)$ is a non-degenerate subspace of $\mathcal{H}_J^2$ implies that its dimension is at least $2k$. Thus $m \ge 2k-1$.

Furthermore, the Gram matrix

$$\Gamma = \left[\gamma_{j,\ell}\right]_{j,\ell=0}^{2k-1}, \quad \gamma_{j,\ell} := \langle \mathbf{h}_\ell, \mathbf{h}_j \rangle_{\mathcal{H}_J^2},$$

associated with the first $2k$ elements of the basis $\{\mathbf{h}_j(z)\}_{j=0}^m$, is given by

$$\Gamma = -(\mathbf{u}^* P_2 \mathbf{u}) \begin{bmatrix} 0 & \Delta^* \\ \Delta & \Delta\Delta^* \end{bmatrix},$$

where Δ denotes the $k \times k$ lower-triangular Toeplitz matrix

$$\Delta := \begin{bmatrix} c_k & 0 & \cdots & 0 \\ c_{k+1} & c_k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{2k-2} & c_{2k-3} & \cdots & 0 \\ c_{2k-1} & c_{2k-2} & \cdots & c_k \end{bmatrix}.$$

Since the matrix Δ is invertible, so is the Gram matrix Γ:

$$\Gamma^{-1} = \frac{1}{\mathbf{u}^* P_2 \mathbf{u}} \begin{bmatrix} I_k & -\Delta^{-1} \\ -\Delta^{-*} & 0 \end{bmatrix}. \tag{3.7}$$

Hence $\operatorname{span}_{\mathbb{C}}\{\mathbf{h}_0(z), \dots, \mathbf{h}_{2k-1}(z)\}$ is a non-degenerate subspace of $\mathcal{H}_J^2$. Since $\Theta(z)$ is an elementary factor in the class $\mathcal{U}_{\mathbf{S}}$, we have $m = 2k-1$ and

$$\dim \mathcal{P}(K_{\mathbf{S}}^\Theta) = 2k.$$

Denote by $q(z)$ the polynomial

$$q(z) := c_k + c_{k+1}z + \cdots + c_{2k-1}z^{k-1}.$$

Then, in view of (3.6), the last element $\mathbf{h}(z) := \mathbf{h}_{2k-1}(z)$ of the basis is given by (3.4), and the space $\mathcal{P}(K_{\mathbf{S}}^\Theta)$ is spanned by $\mathbf{h}(z), R_0\mathbf{h}(z), \dots, R_0^{2k-1}\mathbf{h}(z)$. To determine the form of $\Theta(z)$ in this case, form the polynomial matrix function

$$H(z) := \begin{bmatrix} \mathbf{h}_0(z) & \cdots & \mathbf{h}_{2k-1}(z) \end{bmatrix}$$

and observe that the kernel

$$H(z)\Gamma^{-1}H(w)^*$$

is the reproducing kernel of the space $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$. Thus

$$\frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} = H(z)\Gamma^{-1}H(w)^*. \tag{3.8}$$

Let

$$p(z) = d_0 + d_1 z + \cdots + d_{k-1}z^{k-1}$$

denote the polynomial determined by

$$p(z)q(z) = 1 + \mathrm{O}(z^k), \quad z \to 0.$$

Then its coefficients are the elements of the Toeplitz matrix

$$\Delta^{-1} = \begin{bmatrix} d_0 & 0 & \cdots & 0 \\ d_1 & d_0 & \cdots & 0 \\ \vdots & \vdots & & 0 \\ d_{k-1} & d_{k-2} & \cdots & d_0 \end{bmatrix},$$

and the relations (3.7) and (3.8) imply that

$$J - \Theta(z)J\Theta(w)^* = \left(1 - (zw^*)^k\right)J - \left(z^k\left(p(w) - w^{2k}p^{\#}(w)\right)^* + (w^*)^k\left(p(z) - z^{2k}p^{\#}(z)\right)\right)\frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^*P_2\mathbf{u}}.$$

Setting $w = 1$ in this last identity, we get

$$\Theta(z) = \left(z^k I_2 + \left(p(z) - z^{2k}p^{\#}(z)\right)\frac{\mathbf{u}\mathbf{u}^*J}{\mathbf{u}^*P_2\mathbf{u}}\right)\left(I_2 + \left(p(1) - p(1)^*\right)\frac{\mathbf{u}\mathbf{u}^*J}{\mathbf{u}^*P_2\mathbf{u}}\right)\Theta(1).$$

Thus, $\Theta(z)$ is, up to multiplication from the right by a J-unitary matrix, of the form (3.3). □

3.2. The basic interpolation problem in the class $\mathbf{S}_\kappa$

The solution of the Problem 2.8 in case $\mathbf{u}_0^*J\mathbf{u}_0 \neq 0$ is given by the following theorem. For simplicity we consider the case that $z_1 = 0$.

Theorem 3.4. *Let* $\mathbf{u}_0 \in \mathbb{C}^2$ *and assume that* $\mathbf{u}_0$ *is not* J*-neutral. Denote by* $\Theta(z)$ *the* $\mathbb{C}^{2\times 2}$*-valued function*

$$\Theta(z) := I_2 - (1 - z)\frac{\mathbf{u}_0\mathbf{u}_0^*J}{\mathbf{u}_0^*J\mathbf{u}_0}. \tag{3.9}$$

Then the following statements about a function $s(z)$ *are equivalent.*

(1) $s(z) \in \mathbf{S}_\kappa$ *and any projective representation* $\mathbf{s}(z)$ *of* $s(z)$ *satisfies the condition*

$$\mathbf{u}_0^*J\mathbf{s}(0) = 0. \tag{3.10}$$

(2) $s(z)$ *is of the form*

$$s(z) = T_{\Theta(z)}(\hat{s}(z)),$$

where $\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ *with*

$$\hat{\kappa} := \begin{cases} \kappa, & \mathbf{u}_0^*J\mathbf{u}_0 > 0, \\ \kappa - 1, & \mathbf{u}_0^*J\mathbf{u}_0 < 0, \end{cases} \tag{3.11}$$

and any projective representation $\hat{\mathbf{s}}(z)$ *of* $\hat{s}(z)$ *satisfies the condition*

$$\mathbf{u}_0^{\times *} J\hat{\mathbf{s}}(0) \neq 0. \tag{3.12}$$

Remark 3.5. Theorem 3.4 implies that in the case $\kappa = 0$ and $\mathbf{u}_0^* J\mathbf{u}_0 < 0$ the Problem 2.8 does not have solutions. This is clear since the elements of the Schur class $\mathbf{S}_0$ are analytic and contractive in $\mathbb{D}$; see also the beginning of Section 2.2 and the remarks after Problem 2.8.

Proof of Theorem 3.4. First we recall that according to Theorem 3.2 the matrix function $\Theta(z)$, defined in (3.9), belongs to $\mathcal{U}_{\mathbf{S}}$, and that the reproducing kernel Pontryagin space $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ is one-dimensional:

$$\mathcal{P}(K_{\mathbf{S}}^{\Theta}) = \operatorname{span}_{\mathbb{C}}\{\mathbf{u}_0\}.$$

In view of statement (1) of Theorem 3.1, the kernel $K_{\mathbf{S}}^{\Theta}(z, w)$ is positive in $\mathbb{D}$ if $\mathbf{u}_0^* J\mathbf{u}_0 > 0$, and negative in $\mathbb{D}$ if $\mathbf{u}_0^* J\mathbf{u}_0 < 0$.

Furthermore, according to the same Theorem 3.1, $\Theta(z)$ is J-unitary on the unit circle. Hence $\Theta(z)J\Theta^{\#}(z) = J$, $\det(\Theta(z))$ does not vanish in $\mathbb{C} \setminus \{0\}$, and the matrix polynomial $\Theta(z)^{\times *}$ satisfies the identity

$$\Theta(z)^{\times *} = \det(\Theta(z))\,\Theta^{\#}(z) = \det(\Theta(z))\,J\,\Theta(z)^{-1}J. \tag{3.13}$$

In view of these facts, the equivalence of statements (1) and (2) can be shown as follows.

(1) $\Longrightarrow$ (2). Let $s(z)$ be a function from the class $\mathbf{S}_\kappa$ with the projective representation $\mathbf{s}(z)$, and assume that $\mathbf{s}(z)$ satisfies condition (3.10). Then the vectors $\mathbf{s}(0)^{\times}$ and $\mathbf{u}_0$ are collinear. The relation $\mathbf{u}_0^* J\Theta(0) = 0$ implies $\mathbf{s}(0)^{\times *} J\Theta(0) = 0$; hence $\Theta(0)^{\times *} J\mathbf{s}(0) = 0$. Thus the $\mathbb{C}^2$-valued function

$$\hat{\mathbf{s}}(z) := \frac{J\Theta(z)^{\times *} J\mathbf{s}(z)}{z} \tag{3.14}$$

is analytic in $\mathbb{D}$.

Since $\Theta(z)\mathbf{u}_0 = z\mathbf{u}_0$, it follows that

$$\begin{aligned}\mathbf{u}_0^{\times *} J\hat{\mathbf{s}}(0) &= \lim_{z\to 0} \frac{\mathbf{u}_0^{\times *}\Theta(z)^{\times *} J\mathbf{s}(z)}{z} = \lim_{z\to 0} \mathbf{u}_0^{\times *} J\mathbf{s}(z)\\ &= \mathbf{u}_0^{\times *} J\mathbf{s}(0) = -\mathbf{s}(0)^{\times *} J\mathbf{u}_0 \neq 0.\end{aligned}$$

Hence condition (3.12) is in force; in particular, $\hat{\mathbf{s}}(0) \neq 0$. Furthermore, as follows from (3.13) and (3.14),

$$\Theta(z)\,\hat{\mathbf{s}}(z) = \frac{\det(\Theta(z))}{z}\mathbf{s}(z). \tag{3.15}$$

Thus the vector function $\hat{\mathbf{s}}(z)$ is non-vanishing in $\mathbb{D}$.

We claim that $\hat{\mathbf{s}}(z)$ belongs to the class $\mathbf{S}_{\hat{\kappa}}^{\mathrm{proj}}$, where $\hat{\kappa}$ is given by (3.11). If this fact has been established, it follows that $\hat{\mathbf{s}}(z)$ is a projective representation of a function $\hat{s}(z)$ from the class $\mathbf{S}_{\hat{\kappa}}$ (see Example 2.2) and then statement (1) follows from (3.15).

To prove the claim, write (3.14) as

$$z\hat{\mathbf{s}}(z)^{\times *} J = \mathbf{s}(z)^{\times *} J\Theta(z). \tag{3.16}$$

The equality (3.16) implies that the kernel $K^{\mathbf{s}}_{\mathbf{S}}(z,w)$ can be represented in the form

$$K^{\mathbf{s}}_{\mathbf{S}}(z,w) = K_1(z,w) + K_2(z,w), \tag{3.17}$$

where

$$K_1(z,w) := \mathbf{s}(z)^{\times *} J K^{\Theta}_{\mathbf{S}}(z,w) J \mathbf{s}(w)^{\times}, \qquad K_2(z,w) := z K^{\hat{\mathbf{s}}}_{\mathbf{S}}(z,w) w^*.$$

Since the vectors $\mathbf{s}(0)^{\times}$ and $\mathbf{u}_0$ are collinear,

$$\operatorname{span}_{\mathbb{C}}\{\mathbf{s}(z)^{\times *} J \mathbf{u}_0\} = \operatorname{span}_{\mathbb{C}}\{K^{\mathbf{s}}_{\mathbf{S}}(z,0)\}$$

and

$$\big\langle \mathbf{s}(z)^{\times *} J \mathbf{u}_0, \mathbf{s}(z)^{\times *} J \mathbf{u}_0 \big\rangle_{\mathcal{P}(K^{\mathbf{s}}_{\mathbf{S}})} = \mathbf{u}_0^* J \mathbf{u}_0 = \langle \mathbf{u}_0, \mathbf{u}_0 \rangle_{\mathcal{P}(K^{\Theta}_{\mathbf{S}})},$$

which means that the mapping

$$\mathbf{h}(z) \mapsto \mathbf{s}(z)^{\times *} J \mathbf{h}(z) \tag{3.18}$$

is an isometry from $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ to $\mathcal{P}(K^{\mathbf{s}}_{\mathbf{S}})$. Therefore, the Pontryagin space $\mathcal{P}(K_1)$ is isometrically included in the Pontryagin space $\mathcal{P}(K^{\mathbf{s}}_{\mathbf{S}})$. But then, as follows from (3.17), the Pontryagin space $\mathcal{P}(K_2)$ is also isometrically included in the Pontryagin space $\mathcal{P}(K^{\mathbf{s}}_{\mathbf{S}})$ and, moreover, $\mathcal{P}(K^{\mathbf{s}}_{\mathbf{S}})$ admits the orthogonal direct sum decomposition

$$\mathcal{P}(K^{\mathbf{s}}_{\mathbf{S}}) = \mathcal{P}(K_1) \oplus \mathcal{P}(K_2).$$

Since $\mathcal{P}(K_1)$ is the image of $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ under the mapping (3.18), the number of negative squares of the kernel $K_1(z,w)$ in $\mathbb{D}$ is equal to zero if $\mathbf{u}_0^* J \mathbf{u}_0 > 0$, and to one if $\mathbf{u}_0^* J \mathbf{u}_0 < 0$ (in the latter case κ must be strictly positive). Thus the kernel $K_2(z,w)$ has $\hat{\kappa}$ negative squares in $\mathbb{D}$, where $\hat{\kappa}$ is given in (3.11). Since

$$K^{\hat{\mathbf{s}}}_{\mathbf{S}}(z,w) = \frac{1}{z} K_2(z,w) \frac{1}{w^*},$$

the kernel $K^{\hat{\mathbf{s}}}_{\mathbf{S}}(z,w)$ has $\hat{\kappa}$ negative squares in $\mathbb{D}$ as well.

(2) $\Longrightarrow$ (1). Let $\hat{s}(z)$ be a function of the class $\mathbf{S}_{\hat{\kappa}}$ with $\hat{\kappa}$ given by (3.11), and let $\hat{\mathbf{s}}(z)$ be a projective representation of $\hat{s}(z)$. Assume that $\hat{\mathbf{s}}(z)$ satisfies condition (3.12), then $\Theta(0)\hat{\mathbf{s}}(0) \neq 0$. Therefore, the $\mathbb{C}^2$-valued function $\mathbf{s}(z)$ defined by

$$\mathbf{s}(z) := \Theta(z)\hat{\mathbf{s}}(z) \tag{3.19}$$

is analytic in $\mathbb{D}$ and does not vanish there.

In view of (3.13), the identity (3.19) can be written as

$$\mathbf{s}(z)^{\times *} J \Theta(z) = \det(\Theta(z)) \hat{\mathbf{s}}(z)^{\times *} J.$$

Hence

$$K^{\mathbf{s}}_{\mathbf{S}}(z,w) = \mathbf{s}(z)^{\times *} J K^{\Theta}_{\mathbf{S}}(z,w) J \mathbf{s}(w)^{\times} + \det(\Theta(z)) K^{\hat{\mathbf{s}}}_{\mathbf{S}}(z,w) \det(\Theta(w))^*,$$

and, therefore, the kernel $K^{\mathbf{s}}_{\mathbf{S}}(z,w)$ has finitely many negative squares in $\mathbb{D}$, say κ_0. Thus the vector function $\mathbf{s}(z)$ belongs to the class $\mathbf{S}^{\mathrm{proj}}_{\kappa_0}$ and is a projective representation of a function $s(z)$ from the class $\mathbf{S}_{\kappa_0}$, which, in view of (3.19), is given by

$$s(z) = T_{\Theta(z)}\big(\hat{s}(z)\big).$$

Since

$$\mathbf{u}_0^* J\mathbf{s}(0) = \left(\mathbf{u}_0^* J\,\Theta(0)\right)\,\hat{\mathbf{s}}(0) = 0,$$

statement (1) of the theorem holds true with κ_0 instead of κ. Since, as has already been shown, statement (1) implies statement (2), and since the linear fractional transformation $T_{\Theta(z)}$ is one-to-one,

$$\kappa_0 = \left\{\begin{matrix} \hat{\kappa} & \text{if } \mathbf{u}_0^* J\mathbf{u}_0 > 0 \\ \hat{\kappa}+1 & \text{if } \mathbf{u}_0^* J\mathbf{u}_0 < 0 \end{matrix}\right\} = \kappa. \quad \square$$

If the given vector $\mathbf{u}_0$ in Problem 2.8 is J-neutral the solutions are described by the following theorem. For a discussion about the additional data k and $q(z)$ in the augmented Schur parameter we refer to Section 2.2.

Theorem 3.6. *Let $\mathbf{u}_0 \in \mathbb{C}^2$ be non-zero and J-neutral, let $k \in \mathbb{N}$, and let $q(z)$ be a $\mathbb{C}$-valued polynomial with the properties $q(0) \neq 0$ and $\deg(q) < k$. Denote by $\Theta(z)$ the $\mathbb{C}^{2\times 2}$-valued function*

$$\Theta(z) := z^k I_2 + (p(z) - z^{2k} p^{\#}(z))\frac{\mathbf{u}_0\mathbf{u}_0^* J}{\mathbf{u}_0^* P_2\mathbf{u}_0}, \tag{3.20}$$

where P_2 is defined by (2.6) and $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \text{and} \quad p(z)q(z) = 1 + \mathrm{O}(z^k), \quad z \to 0.$$

Then the following statements about a function $s(z)$ are equivalent.

(1) *$s(z) \in \mathbf{S}_\kappa$ and any projective representation $\mathbf{s}(z)$ of $s(z)$ satisfies the condition*

$$\frac{\mathbf{u}_0^* J\mathbf{s}(z)}{\mathbf{u}_0^* P_2\mathbf{s}(z)} = z^k q(z) + \mathrm{O}(z^{2k}), \quad z \to 0. \tag{3.21}$$

(2) *$s(z)$ is of the form*

$$s(z) = T_{\Theta(z)}(\hat{s}(z)),$$

where $\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ with $\hat{\kappa} \in \mathbb{N}_0$ given by

$$\hat{\kappa} := \kappa - k, \tag{3.22}$$

and any projective representation $\hat{\mathbf{s}}(z)$ of $\hat{s}(z)$ satisfies the condition

$$\mathbf{u}_0^{\times *} J\hat{\mathbf{s}}(0) \neq 0. \tag{3.23}$$

Remark 3.7. Theorem 3.6 implies, in particular, that in the case $\mathbf{u}_0^* J\mathbf{u}_0 = 0$ every non-constant solution $s(z)$ of Problem 2.8 has the property that the order k of the zero of $s(z) - s(0)$ at $z = 0$ does not exceed κ; in other words, if $k > \kappa$ there is no solution.

Proof of Theorem 3.6. The proof parallels that of Theorem 3.4. In view of Theorem 3.2, the matrix function $\Theta(z)$, defined in (3.20), belongs to $\mathcal{U}_{\mathbf{S}}$ and the kernel $K_{\mathbf{S}}^{\Theta}(z, w)$ has k positive and k negative squares in $\mathbb{D}$. The associated reproducing kernel Pontryagin space $\mathcal{P}(K^{\Theta}_{\mathbf{S}})$ can be characterized as follows:

$$\mathcal{P}(K^{\Theta}_{\mathbf{S}}) = \operatorname{span}_{\mathbb{C}} \left\{ \mathbf{h}(z), R_0\mathbf{h}(z), \ldots, R_0^{2k-1}\mathbf{h}(z) \right\},$$

where $\mathbf{h}(z)$ is the $\mathbb{C}^2$-valued polynomial given by

$$\mathbf{h}(z) := \left(z^{2k-1} I_2 + z^{k-1} q^{\#}(z) P_2 \right) \mathbf{u}_0.$$

Furthermore, since $\Theta(z)$ is J-unitary on the unit circle, $\det(\Theta(z))$ does not vanish in $\mathbb{C} \setminus \{0\}$ and the matrix polynomial $\Theta(z)^{\times *}$ satisfies the identity (3.13). In view of these facts, the equivalence of statements (1) and (2) can be shown as follows.

(1) $\Longrightarrow$ (2). Let $s(z) \in \mathbf{S}_\kappa$ with the projective representation $\mathbf{s}(z)$. Assume that $\mathbf{s}(z)$ satisfies (3.21). Then there exists a polynomial $g(z)$ with the properties $g(0) \neq 0$, $\deg(g) \leq 2k + 1$ and

$$\mathbf{s}(z) = g(z) \left(I_2 + z^k q(z)(I_2 - P_2) \right) \mathbf{u}_0^{\times} + \mathrm{O}(z^{2k}), \quad z \to 0.$$

Hence

$$\mathbf{s}(z)^{\times *} = g(z) z^{2k-1} \mathbf{h}^{\#}(z) + \mathrm{O}(z^{2k}), \quad z \to 0. \tag{3.24}$$

Since, according to statement (4) of Theorem 3.1,

$$\lim_{z \to 0} \mathbf{h}^{\#}(z) J \Theta(z) = 0,$$

we have

$$\lim_{z \to 0} \frac{\mathbf{s}(z)^{\times *} J \Theta(z)}{z^{2k-1}} = 0$$

and hence the $\mathbb{C}^2$-valued function

$$\hat{\mathbf{s}}(z) := \frac{J \Theta(z)^{\times *} J \mathbf{s}(z)}{z^{2k}} \tag{3.25}$$

is analytic in $\mathbb{D}$.

The relation $\Theta(z)\mathbf{u}_0 = z^k \mathbf{u}_0$ implies that

$$\begin{aligned} \mathbf{u}_0^{\times *} J \hat{\mathbf{s}}(0) &= \lim_{z \to 0} \frac{\mathbf{u}_0^{\times *} \Theta(z)^{\times *} J \mathbf{s}(z)}{z^{2k}} = \lim_{z \to 0} \frac{\mathbf{u}_0^{\times *} J \mathbf{s}(z)}{z^k} \\ &= \lim_{z \to 0} g(z) \frac{\mathbf{u}_0^{\times *} J (I_2 + z^k q(z)(I_2 - P_2)) \mathbf{u}_0^{\times}}{z^k} = -g(0) q(0) \mathbf{u}_0^{\times *} P_2 \mathbf{u}_0^{\times} \neq 0. \end{aligned}$$

Hence condition (3.23) is in force; in particular, $\hat{\mathbf{s}}(0) \neq 0$. Furthermore, as follows from (3.13) and (3.25),

$$\Theta(z)\, \hat{\mathbf{s}}(z) = \frac{\det(\Theta(z))}{z^{2k}} \mathbf{s}(z). \tag{3.26}$$

Thus the vector function $\hat{\mathbf{s}}(z)$ is non-vanishing in $\mathbb{D}$. We claim that the vector function $\hat{\mathbf{s}}(z)$ belongs to the class $\mathbf{S}^{\mathrm{proj}}_{\hat{\kappa}}$, where $\hat{\kappa}$ is given by (3.22). The claim implies that $\hat{\mathbf{s}}(z)$ is a projective representation of a function $\hat{s}(z)$ from the class $\mathbf{S}_{\hat{\kappa}}$ (see Example 2.2) and then statement (1) follows from (3.26).

To prove the claim we write (3.25) as

$$z^{2k}\hat{\mathbf{s}}(z)^{\times *}J = \mathbf{s}(z)^{\times *}J\,\Theta(z). \tag{3.27}$$

The equality (3.27) implies that the kernel $K_{\mathbf{S}}^{\mathbf{s}}(z,w)$ can be represented in the form

$$K_{\mathbf{S}}^{\mathbf{s}}(z,w) = K_1(z,w) + K_2(z,w), \tag{3.28}$$

where

$$K_1(z,w) := \mathbf{s}(z)^{\times *}JK_{\mathbf{S}}^{\Theta}(z,w)J\mathbf{s}(w)^{\times}, \qquad K_2(z,w) := z^{2k}K_{\mathbf{S}}^{\hat{\mathbf{s}}}(z,w)(w^{2k})^*.$$

Since the kernel $K_{\mathbf{S}}^{\Theta}(z,w)$ has finitely many positive and negative squares in $\mathbb{D}$, so does the kernel $K_1(z,w)$. Hence the kernel $K_2(z,w)$ has finitely many negative squares in $\mathbb{D}$. Further, in view of (3.24), for $j,\ell = 0,1,\dots,2k-1$ it holds that

$$\mathbf{s}(z)^{\times *}J\left(R_0^{2k-1-j}\mathbf{h}\right)(z) = \frac{1}{j!}\frac{\partial^j}{(\partial w^*)^j}\frac{K_{\mathbf{S}}^{\mathbf{s}}(z,w)}{g(w)^*}\Bigg|_{w=0},$$

$$\langle \mathbf{s}^{\times *}J(R_0^{2k-1-j}\mathbf{h}), \mathbf{s}^{\times *}J(R_0^{2k-1-\ell}\mathbf{h})\rangle_{\mathcal{P}(K_{\mathbf{S}}^{\mathbf{s}})} = \frac{1}{j!\ell!}\frac{\partial^{j+\ell}}{(\partial z)^\ell(\partial w^*)^j}\frac{K_{\mathbf{S}}^{\mathbf{s}}(z,w)}{g(z)g(w)^*}\Bigg|_{\substack{z=0\\w=0}}$$

$$= \frac{1}{j!\ell!}\frac{\partial^{j+\ell}}{(\partial z)^\ell(\partial w^*)^j}\frac{z^{2k-1}(w^*)^{2k-1}\mathbf{h}^\#(z)J\mathbf{h}^\#(w)^*}{1-zw^*}\Bigg|_{\substack{z=0\\w=0}}$$

$$= \left\langle R_0^{2k-1-j}\mathbf{h}, R_0^{2k-1-\ell}\mathbf{h}\right\rangle_{\mathcal{P}(K_{\mathbf{S}}^{\Theta})},$$

which means that the mapping

$$\mathbf{f}(z) \mapsto \mathbf{s}(z)^{\times *}J\mathbf{f}(z) \tag{3.29}$$

is an isometry from $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ to $\mathcal{P}(K_{\mathbf{S}}^{\mathbf{s}})$. Therefore, the Pontryagin space $\mathcal{P}(K_1)$ is isometrically included in the Pontryagin space $\mathcal{P}(K_{\mathbf{S}}^{\mathbf{s}})$. But then, as follows from the identity (3.28), the Pontryagin space $\mathcal{P}(K_2)$ is also isometrically included in the Pontryagin space $\mathcal{P}(K_{\mathbf{S}}^{\mathbf{s}})$ and, moreover, $\mathcal{P}(K_{\mathbf{S}}^{\mathbf{s}})$ admits the orthogonal direct sum decomposition

$$\mathcal{P}(K_{\mathbf{S}}^{\mathbf{s}}) = \mathcal{P}(K_1) \oplus \mathcal{P}(K_2).$$

Since $\mathcal{P}(K_1)$ is the image of $\mathcal{P}(K_{\mathbf{S}}^{\Theta})$ under the mapping (3.29), the kernel $K_1(z,w)$ has k positive and k negative squares in $\mathbb{D}$ (and hence $\kappa \geq k$). Thus the kernel $K_2(z,w)$ has $\hat{\kappa}$ negative squares in $\mathbb{D}$, where $\hat{\kappa}$ is given by (3.22). Since

$$K_{\mathbf{S}}^{\hat{\mathbf{s}}}(z,w) = \frac{1}{z^{2k}}K_2(z,w)\frac{1}{(w^*)^{2k}},$$

the kernel $K_{\mathbf{S}}^{\hat{\mathbf{s}}}(z,w)$ has $\hat{\kappa}$ negative squares in $\mathbb{D}$ as well.

(2) $\Longrightarrow$ (1). Let $\hat{s}(z)$ be a function from the class $\mathbf{S}_{\hat{\kappa}}$ with $\hat{\kappa}$ given by (3.22), and let $\hat{\mathbf{s}}(z)$ be a projective representation of $\hat{s}(z)$. Assume that $\hat{\mathbf{s}}(z)$ satisfies condition (3.23). Then $\Theta(0)\,\hat{\mathbf{s}}(0) \neq 0$ and therefore the $\mathbb{C}^2$-valued function $\mathbf{s}(z)$ defined by

$$\mathbf{s}(z) := \Theta(z)\,\hat{\mathbf{s}}(z) \tag{3.30}$$

is analytic in $\mathbb{D}$ and does not vanish there. In view of (3.13), the identity (3.30) can be re-written as

$$\mathbf{s}(z)^{\times *} J\,\Theta(z) = \det(\Theta(z))\,\hat{\mathbf{s}}(z)^{\times *} J.$$

Hence

$$K_{\mathbf{S}}^{\mathbf{s}}(z,w) = \mathbf{s}(z)^{\times *} J K_{\mathbf{S}}^{\Theta}(z,w) J\mathbf{s}(w)^{\times} + \det(\Theta(z)) K_{\mathbf{S}}^{\hat{\mathbf{s}}}(z,w)\det(\Theta(w))^*,$$

and, therefore, the kernel $K_{\mathbf{S}}^{\mathbf{s}}(z,w)$ has finitely many negative squares in $\mathbb{D}$, say κ_0. Thus the vector function $\mathbf{s}(z)$ belongs to the class $\mathbf{S}_{\kappa_0}^{\mathrm{proj}}$ and is a projective representation of a function $s(z)$ from the class $\mathbf{S}_{\kappa_0}$, which, in view of (3.30), is given by

$$s(z) = T_{\Theta(z)}\left(\hat{s}(z)\right).$$

Since

$$\mathbf{u}_0^* J\mathbf{s}(z) = \mathbf{u}_0^* J\,\Theta(z)\hat{\mathbf{s}}(z) = z^k \mathbf{u}_0^* J\hat{\mathbf{s}}(z),$$

and

$$\mathbf{u}_0^* P_2\mathbf{s}(z) = \mathbf{u}_0^* P_2\,\Theta(z)\hat{\mathbf{s}}(z) = p(z)\mathbf{u}_0^* J\hat{\mathbf{s}}(z) + \mathrm{O}(z^k), \quad z \to 0,$$

condition (3.21) is in force. Thus statement (1) of the theorem holds true with κ_0 instead of κ. Since, as was already shown, statement (1) implies statement (2), and since the linear fractional transformation $T_{\Theta(z)}$ is one-to-one,

$$\kappa_0 = \hat{\kappa} + k = \kappa. \quad \square$$

Remark 3.8. In the description of Theorem 3.4(2) and Theorem 3.6(2) of the solutions of Problem 2.8 by the linear fractional transformation $s(z) = T_{\Theta(z)}(\hat{s}(z))$ one can replace $\Theta(z)$ with $\Theta(z)N$ where $N \in \mathbb{C}^{2\times 2}$ is an arbitrary J-unitary constant matrix. Then Theorems 3.4 and 3.6 still hold with conditions (3.12) and (3.23) replaced by

$$\mathbf{u}_0^{\times *} J N\hat{\mathbf{s}}(0) \neq 0.$$

3.3. *The Schur transformation in* $\mathbf{S}_\kappa$

The linear fractional transformation

$$s(z) = T_{\Theta(z)}(\hat{s}(z)),$$

in Theorems 3.4 and 3.6, establishes a one-to-one correspondence between the parameters $\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ and the solutions $s(z) \in \mathbf{S}_\kappa$ of Problem 2.8. The inverse correspondence is the Schur transformation centered at $z_1 = 0$ in the class $\mathbf{S}_\kappa$. It is defined as follows.

Definition 3.9. Let $s(z) \in \mathbf{S}_\kappa$ with a projective representation $\mathbf{s}(z)$, and denote by $\mathbf{u}$ the vector

$$\mathbf{u} := \mathbf{s}(0)^{\times}.$$

If

$$\mathbf{u}^{\times *} J\mathbf{s}(z) \not\equiv 0, \tag{3.31}$$

then the Schur transform $\hat{s}(z)$ of $s(z)$, centered at $z_1 = 0$, is

$$\hat{s}(z) := T_{\Phi(z)}(s(z)),$$

where $\Phi(z)$ is defined as follows.

(1) If $\mathbf{u}^*J\mathbf{u} \neq 0$, then

$$\Phi(z) := zI_2 + (1-z)\frac{\mathbf{u}\mathbf{u}^*J}{\mathbf{u}^*J\mathbf{u}}. \tag{3.32}$$

(2) If $\mathbf{u}^*J\mathbf{u} = 0$, then

$$\Phi(z) := z^k I_2 - \left(p(z) - z^{2k}p^{\#}(z)\right)\frac{\mathbf{u}\mathbf{u}^*J}{\mathbf{u}^*P_2\mathbf{u}}, \tag{3.33}$$

where k is the order of the zero of $\mathbf{u}^*J\mathbf{s}(z)$ at $z=0$ and $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \text{and} \quad \frac{z^k\mathbf{u}^*P_2\mathbf{s}(z)}{\mathbf{u}^*J\mathbf{s}(z)} = p(z) + \mathrm{O}(z^k), \quad z \to 0.$$

Remark 3.10. Condition (3.31) is equivalent to

$$K^s_{\mathbf{S}}(z,w) \not\equiv 0.$$

If the vector $\mathbf{u}$ is J-neutral, it is also equivalent to

$$\mathbf{u}^*J\mathbf{s}(z) \not\equiv 0.$$

Remark 3.11. The polynomial $\Phi(z)$ defined in (3.32) is related to the polynomial $\Theta(z)$ from (3.9) by

$$\Phi(z) = z\Theta(z)^{-1},$$

while the polynomial $\Phi(z)$ in (3.33) and the polynomial $\Theta(z)$ in (3.20) are related by

$$\Phi(z) = z^{2k}\Theta(z)^{-1}.$$

Thus, in both cases,

$$T_{\Phi(z)} = T^{-1}_{\Theta(z)}.$$

Remark 3.12. Definition 3.9 of the Schur transformation centered at 0 in the class of generalized Schur functions differs somewhat from the classical definition, which can be found, for instance, in [38]. In view of Remark 3.8, however, the polynomial $\Phi(z)$ in Definition 3.9 can be replaced by $N\Phi(z)$ where $N \in \mathbb{C}^{2\times 2}$ is an arbitrary J-unitary matrix. Thus various equivalent versions of the Schur transformation can be defined, including the one in [38]. We explain this in more detail.

If $s(z)$ is analytic at $z=0$ and $|s(0)| \neq 1$, the matrix function $\Phi(z)$ in (3.32) is

$$\Phi(z) = \frac{1}{1-|s(0)|^2}\begin{bmatrix} 1 - z|s(0)|^2 & (z-1)s(0) \\ (1-z)s(0)^* & z - |s(0)|^2 \end{bmatrix}. \tag{3.34}$$

If $|s(0)| < 1$, the matrix function $\Phi(z)$ from (3.34) can be replaced by $N_1\Phi(z)$, where N_1 denotes the J-unitary matrix

$$N_1 := \frac{1}{\sqrt{1-|s(0)|^2}}\begin{bmatrix} 1 & -s(0) \\ -s(0)^* & 1 \end{bmatrix}.$$

Then

$$N_1 \Phi(z) = \frac{1}{\sqrt{1-|s(0)|^2}} \begin{bmatrix} 1 & -s(0) \\ -zs(0)^* & z \end{bmatrix},$$

and

$$T_{N_1 \Phi(z)}(s(z)) = \frac{s(z)-s(0)}{z\,(1-s(z)s(0)^*)}. \tag{3.35}$$

If $|s(0)| > 1$, $\Phi(z)$ from (3.34) can be replaced by $N_2 \Phi(z)$, where N_2 denotes the J-unitary matrix

$$N_2 := \frac{1}{\sqrt{|s(0)|^2-1}} \begin{bmatrix} -s(0)^* & 1 \\ 1 & -s(0) \end{bmatrix}.$$

Then

$$N_2 \Phi(z) = \frac{1}{\sqrt{1-|s(0)|^2}} \begin{bmatrix} 1 & -s(0) \\ -zs(0)^* & z \end{bmatrix},$$

and we obtain again

$$T_{N_2 \Phi(z)}(s(z)) = \frac{s(z)-s(0)}{z\,(1-s(z)s(0)^*)}.$$

Furthermore, in the case when $s(z)$ is analytic at 0 and $|s(0)| = 1$ (but $|s(z)| \not\equiv 1$), the matrix function $\Phi(z)$ in (3.33) is of the form

$$\Phi(z) = -\begin{bmatrix} \left(p(z) - z^{2k} p^{\#}(z)\right) - z^k & -\left(p(z) - z^{2k} p^{\#}(z)\right) s(0) \\ \left(p(z) - z^{2k} p^{\#}(z)\right) s(0)^* & -\left(p(z) - z^{2k} p^{\#}(z)\right) - z^k \end{bmatrix},$$

where k is the order of the zero of $s(z) - s(0)$ at $z = 0$, and $p(z)$ is the polynomial determined by

$$\frac{z^k s(0)}{s(z)-s(0)} = p(z) + \mathrm{O}(z^k) \quad \text{as } z \to 0.$$

Hence

$$T_{\Phi(z)}(s(z)) = \frac{\left(p(z) - z^{2k} p^{\#}(z) - z^k\right) s(z) - \left(p(z) - z^{2k} p^{\#}(z)\right) s(0)}{\left(p(z) - z^{2k} p^{\#}(z)\right) s(0)^* s(z) - \left(p(z) - z^{2k} p^{\#}(z)\right) - z^k}.$$

Finally, in the case when $s(z)$ has a pole at $z = 0$, the matrix function $\Phi(z)$ in (3.32) is of the form

$$\Phi(z) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$T_{\Phi(z)}(s(z)) = zs(z). \tag{3.36}$$

The formulas (3.35)–(3.36) coincide with the definition of the Schur transformation in [6].

4. A characterization of the classes $\widetilde{\Sigma}_\kappa(Q,\rho)$ and $\mathcal{U}(Q,\rho)$ in terms of generalized Schur functions

The aim of this section is to show that the elements of the classes $\widetilde{\Sigma}_\kappa(Q,\rho)$ and $\mathcal{U}(Q,\rho)$ can be expressed in terms of the generalized Schur functions, considered in Section 3. The main result is Theorem 4.18, which will be stated and proved after some preparation.

4.1. Reduction to the case $Q = I_2$

Proposition 4.1. *The class $\widetilde{\Sigma}_\kappa(Q,\rho)$ can be characterized as follows:*

$$\widetilde{\Sigma}_\kappa(Q,\rho) = \left\{T_{Q(z)}(f(z)) : f(z) \in \Sigma_\kappa(I_2,\rho)\right\}.$$

Proof. Suppose that $\mathbf{f}(z)$ and $\mathbf{g}(z)$ are two $\mathbb{C}^2$-valued functions, which are analytic in $\Omega_+(\rho)$, do not vanish there and satisfy the relation

$$\mathbf{g}(z)\, q(z) = Q(z)\, \mathbf{f}(z), \quad z \in \mathrm{hol}\,(g, Q),$$

where $q(z)$ is a complex function, meromorphic in $\Omega_+(\rho)$. Then

$$q(z)\mathbf{g}(z)^{\times *} J\, Q(z) = \det(Q(z))\mathbf{f}(z)^{\times *} J,$$

and hence

$$q(z) K^{\mathbf{g}}_{Q,\rho}(z,w) q(w)^* = \det(Q(z)) K^{\mathbf{f}}_{I_2,\rho}(z,w) \det(Q(w))^*.$$

Thus the kernels $K^{\mathbf{g}}_{Q,\rho}(z,w)$ and $K^{\mathbf{f}}_{I_2,\rho}(z,w)$ have the same number of negative squares in $\Omega_+(\rho)$. In view of Definition 2.4 and Example 2.5, this completes the proof. $\square$

Proposition 4.2. *The class $\mathcal{U}(Q,\rho)$ can be characterized as follows:*

$$\mathcal{U}(Q,\rho) = \left\{Q(z)\Theta(z)Q(z)^{-1} : \Theta(z) \in \mathcal{U}(I_2,\rho)\right\}.$$

Proof. In view of Definition 2.12, it suffices to show that for any two $\mathbb{C}^{2\times 2}$-valued functions $\Phi(z)$ and $\Theta(z)$, which are meromorphic in Ω and satisfy the relation

$$Q(z)\Theta(z) = \Phi(z) Q(z), \tag{4.1}$$

the kernels $K^{\Phi}_{Q,\rho}(z,w)$ and $K^{\Theta}_{I_2,\rho}(z,w)$ have the same number of positive and of negative squares in $\Omega_+(\rho)$. This follows immediately from the observation that (4.1) implies

$$K^{\Phi}_{Q,\rho}(z,w) = Q(z) K^{\Theta}_{I_2,\rho}(z,w) Q(w)^*. \quad \square$$

4.2. A class of operator-valued functions

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two Pontryagin spaces with the same index. Denote by $L(\mathcal{P}_1,\mathcal{P}_2)$ the space of continuous linear operators from $\mathcal{P}_1$ to $\mathcal{P}_2$, and by $^{[*]}$ the adjoint with respect to the indefinite inner products in $\mathcal{P}_1$ and $\mathcal{P}_2$.

Definition 4.3. Let Ω' be a nonempty open subset of $\Omega_+(\rho)$. An $L(\mathcal{P}_1,\mathcal{P}_2)$-valued function $F(z)$ is said to belong to the class $\mathbf{O}^{\Omega'}_\kappa(\rho,\mathcal{P}_1,\mathcal{P}_2)$ if it is analytic in Ω' and the kernel

$$K_\rho^F(z,w) := \frac{I_{\mathcal{P}_2} - F(z)F(w)^{[*]}}{\rho(z,w)}, \quad z, w \in \Omega',$$

has κ negative squares.

Example 4.4. Let Ω' be a nonempty open subset of $\mathbb{D}$. According to Definition 4.3, an $L(\mathcal{P}_1, \mathcal{P}_2)$-valued function $S(z)$ belongs to the class $\mathbf{O}_\kappa^{\Omega'}(\rho_{\mathbf{S}}, \mathcal{P}_1, \mathcal{P}_2)$ if and only if it is analytic in Ω' and the kernel

$$K_{\rho_{\mathbf{S}}}^S(z,w) = \frac{I_{\mathcal{P}_2} - S(z)S(w)^{[*]}}{1 - zw^*}, \quad z, w \in \Omega',$$

has κ negative squares. In this case $S(z)$ is an operator-valued generalized Schur function; it admits a meromorphic extension to $\mathbb{D}$ and the associated kernel $K_{\rho_{\mathbf{S}}}^S(z,w)$ has the same number κ of negative squares in any non-empty open subset of $\mathbb{D}$ (see [12, Theorem 0.2]). We use the following notation:

$$\mathbf{S}_\kappa^{\Omega'}(\mathcal{P}_1, \mathcal{P}_2) := \mathbf{O}_\kappa^{\Omega'}(\rho_{\mathbf{S}}, \mathcal{P}_1, \mathcal{P}_2), \qquad \mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2) = \bigcup_{\Omega'} \mathbf{S}_\kappa^{\Omega'}(\mathcal{P}_1, \mathcal{P}_2),$$

where the union is taken over all open subsets Ω' of $\mathbb{D}$. Note that the class $\mathbf{S}_\kappa^{\Omega'}(\mathcal{P}_1, \mathcal{P}_2)$ consists of all elements of $\mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2)$ which are analytic in Ω', and that the class $\mathbf{S}_\kappa(\mathbb{C}, \mathbb{C})$ coincides with $\mathbf{S}_\kappa$.

In order to characterize the class $\mathbf{O}_\kappa^{\Omega'}(\rho, \mathcal{P}_1, \mathcal{P}_2)$ in terms of operator-valued generalized Schur functions, we introduce an auxiliary function, associated with the kernel $\rho(z,w)$.

Definition 4.5. For $z_1 \in \Omega_+(\rho)$ and $z_0 \in \Omega_0(\rho)$ set

$$\Omega^{z_1}(\rho) := \{z \in \Omega : \rho(z, z_1) \neq 0\}$$

and

$$\sigma_\rho^{z_0}(z, z_1) := 1 - \frac{\rho(z_1, z_1)\rho(z, z_0)}{\rho(z, z_1)\rho(z_1, z_0)}, \quad z \in \Omega^{z_1}(\rho). \tag{4.2}$$

Clearly, $\sigma_\rho^{z_0}(z, z_1)$ is a meromorphic function of z in Ω and analytic in $\Omega^{z_1}(\rho)$.

Example 4.6. If $\rho(z,w) = \rho_{\mathbf{S}}(z,w),\ z_1 \in \mathbb{D}$, and $z_0 \in \mathbb{T}$, then

$$\sigma_{\rho_{\mathbf{S}}}^{z_0}(z, z_1) = \sigma_{\mathbf{S}}^{z_0}(z, z_1) = \frac{(z - z_1)(1 - z_0 z_1^*)}{(1 - z z_1^*)(z_0 - z_1)}.$$

The function $\sigma_{\mathbf{S}}^{z_0}(z, z_1)$ maps $\mathbb{D}$ conformally onto itself.

Example 4.7. If $\rho(z,w) = \rho_{\mathbf{N}}(z,w),\ z_1 \in \mathbb{C}_+$, and $z_0 \in \mathbb{R}$, then

$$\sigma_{\rho_{\mathbf{N}}}^{z_0}(z, z_1) = \sigma_{\mathbf{N}}^{z_0}(z, z_1) := \frac{(z - z_1)(z_0 - z_1^*)}{(z - z_1^*)(z_0 - z_1)}.$$

The function $\sigma_{\mathbf{N}}^{z_0}(z, z_1)$ maps $\mathbb{C}_+$ conformally onto $\mathbb{D}$.

Lemma 4.8. *For $z_1 \in \Omega_+(\rho)$ and $z_0 \in \Omega_0(\rho)$ we have*

$$1 - \sigma_\rho^{z_0}(z, z_1)\sigma_\rho^{z_0}(w, z_1)^* = \frac{\rho(z_1, z_1)\rho(z, w)}{\rho(z, z_1)\rho(z_1, w)}, \quad z, w \in \Omega^{z_1}(\rho).$$

Proof. Choose a representation (1.1) of the kernel $\rho(z, w)$ and denote by $\mathbf{a}(z)$ the $\mathbb{C}^2$-valued function

$$\mathbf{a}(z) := \begin{bmatrix} \alpha(z) \\ \beta(z) \end{bmatrix},$$

which is analytic in Ω and does not vanish there. Then

$$\rho(z, w) = \mathbf{a}(w)^* J\mathbf{a}(z) = -\mathbf{a}(z)^{\times *} J\mathbf{a}(w)^{\times}.$$

For any $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ the following relation is easy to check:

$$\mathbf{ab}^{\times *} - \mathbf{ba}^{\times *} = \mathbf{b}^{\times *} J\mathbf{a}J.$$

Therefore, for every pair of points $z, w \in \Omega$ we find

$$\begin{aligned} &\rho(z, z_1)\rho(z_1, w) - \rho(z_1, z_1)\rho(z, w) \\ &\quad = -\mathbf{a}(z_1)^* J\mathbf{a}(z)\mathbf{a}(z_1)^{\times *} J\mathbf{a}(w)^{\times} + \mathbf{a}(z_1)^* J\mathbf{a}(z_1)\mathbf{a}(z)^{\times *} J\mathbf{a}(w)^{\times} \\ &\quad = -\mathbf{a}(z_1)^* J \left(\mathbf{a}(z)\mathbf{a}(z_1)^{\times *} - \mathbf{a}(z_1)\mathbf{a}(z)^{\times *}\right) J\mathbf{a}(w)^{\times} \\ &\quad = -\mathbf{a}(z_1)^* J \left(\mathbf{a}(z_1)^{\times *} J\mathbf{a}(z)J\right) J\mathbf{a}(w)^{\times} \\ &\quad = -\left(\mathbf{a}(z_1)^{\times *} J\mathbf{a}(z)\right) \mathbf{a}(z_1)^* J\mathbf{a}(w)^{\times} \\ &\quad = -\delta(z, z_1)\delta(z_1, w)^* \\ &\quad = \delta(z, z_1)\delta(w, z_1)^*, \end{aligned} \tag{4.3}$$

where

$$\delta(z, z_1) := \mathbf{a}(z_1)^{\times *} J\mathbf{a}(z) = -\delta(z_1, z).$$

Setting $z = w = z_0$ in (4.3) and taking into account that $\rho(z_0, z_0) = 0$, $\rho(z_1, z_0) \neq 0$, it follows that

$$|\rho(z_1, z_0)|^2 = |\delta(z_0, z_1)|^2 \neq 0. \tag{4.4}$$

If $z \in \Omega^{z_1}$ and $w = z_0$, (4.3) and (4.2) imply that

$$\rho(z, z_1)\rho(z_1, z_0)\sigma_\rho^{z_0}(z, z_1) = \delta(z, z_1)\delta(z_0, z_1)^*;$$

similarly,

$$\rho(w, z_1)\rho(z_1, z_0)\sigma_\rho^{z_0}(w, z_1) = \delta(w, z_1)\delta(z_0, z_1)^*.$$

Thus, for every pair of points z, w in Ω^{z_1} it holds that

$$\rho(z, z_1)\rho(z_1, w)|\rho(z_1, z_0)|^2\sigma_\rho^{z_0}(z, z_1)\sigma_\rho^{z_0}(w, z_1)^* = \delta(z, z_1)\delta(w, z_1)^*|\delta(z_0, z_1)|^2.$$

Taking into account (4.4) we find that

$$\rho(z, z_1)\rho(z_1, w)\sigma_\rho^{z_0}(z, z_1)\sigma_\rho^{z_0}(w, z_1)^* = \delta(z, z_1)\delta(w, z_1)^*.$$

Comparison of this relation with (4.3) completes the proof. □

Lemma 4.9. *Let $z_1, z_2 \in \Omega_+(\rho)$, $z_0, z_3 \in \Omega_0(\rho)$, and set*

$$w_j := \sigma_\rho^{z_0}(z_j, z_1) \quad j = 0, 1, 2, 3.$$

Then

$$w_1 = 0,\ w_2 \in \mathbb{D},\ w_0 = 1,\ w_3 \in \mathbb{T}, \tag{4.5}$$

and

$$\sigma_{\mathbf{S}}^{w_3}\left(\sigma_\rho^{z_0}(z, z_1), w_2\right) = \sigma_\rho^{z_3}(z, z_2). \tag{4.6}$$

Proof. Choose a representation (1.1) of the kernel $\rho(z, w)$ and denote by $\varphi(z)$ the complex function

$$\varphi(z) := \frac{\beta(z)}{\alpha(z)},$$

which is meromorphic in Ω. Note that the set of poles of the function $\varphi(z)$ is a subset of $\Omega_-(\rho)$, and that $\varphi(z)$ maps $\Omega_+(\rho)$ into $\mathbb{D}$ and $\Omega_0(\rho)$ into $\mathbb{T}$. Further, the following identity holds:

$$\sigma_\rho^{z_0}(z, z_1) = \sigma_{\mathbf{S}}^{\varphi(z_0)}(\varphi(z), \varphi(z_1)). \tag{4.7}$$

In view of Example 4.6 and the relation

$$\sigma_{\mathbf{S}}^{w_3}(\sigma_{\mathbf{S}}^{\varphi(z_0)}(z, \varphi(z_1)), w_2) = \sigma_{\mathbf{S}}^{\varphi(z_3)}(z, \varphi(z_2)),$$

(4.7) implies (4.5) and (4.6). □

Theorem 4.10. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Pontryagin spaces with the same index, let Ω' be an open subset of $\Omega_+(\rho)$ and $z_1 \in \Omega'$, $z_0 \in \Omega_0(\rho)$. Denote by Ω'' the image of Ω' under the map $\sigma_\rho^{z_0}(z, z_1)$:*

$$\Omega'' := \left\{\sigma_\rho^{z_0}(z, z_1) : z \in \Omega'\right\}.$$

For an $L(\mathcal{P}_1, \mathcal{P}_2)$-valued function $F(z)$ the following statements are equivalent.

(1) *$F(z) \in \mathbf{O}_\kappa^{\Omega'}(\rho, \mathcal{P}_1, \mathcal{P}_2)$.*

(2) *$F(z)$ is of the form*

$$F(z) = S(\sigma_\rho^{z_0}(z, z_1)), \tag{4.8}$$

with $S(z) \in \mathbf{S}_\kappa^{\Omega''}(\mathcal{P}_1, \mathcal{P}_2)$.

The proof of Theorem 4.10 involves an explicit construction of the Schur function $S(z)$, the so-called backward-shift realization. This construction is based on the following lemma, adapted from [10] to the present setting.

Lemma 4.11. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Pontryagin spaces with the same index. Let Ω' be an open subset of $\Omega_+(\rho)$ and $z_1 \in \Omega'$, $z_0 \in \Omega_0(\rho)$. Finally, let $F(z) \in \mathbf{O}_\kappa^{\Omega'}(\rho, \mathcal{P}_1, \mathcal{P}_2)$. If for $p \in \mathcal{P}_1$*

and $f(z) \in \mathcal{P}(K_\rho^F)$ *we define*

$$(Af)(z) = \frac{\rho(z, z_1) f(z) - \rho(z_1, z_1) f(z_1)}{\rho(z, z_1)\sigma_\rho^{z_0}(z, z_1)}, \tag{4.9}$$

$$(Bp)(z) = \sqrt{\rho(z_1, z_1)} \frac{F(z) - F(z_1)}{\rho(z, z_1)\sigma_\rho^{z_0}(z, z_1)} p, \tag{4.10}$$

$$Cf = \sqrt{\rho(z_1, z_1)} f(z_1), \tag{4.11}$$

$$Dp = F(z_1) p, \tag{4.12}$$

then the operator

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{P}(K_\rho^F) \\ \mathcal{P}_1 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{P}(K_\rho^F) \\ \mathcal{P}_2 \end{bmatrix}$$

is coisometric.

Proof. Denote by Ω_1' the set

$$\Omega_1' := \{z \in \Omega' : \sigma_\rho^{z_0}(z, z_1) \neq 0\}$$

and consider the linear relation $\mathcal{R}$ in $(\mathcal{P}(K_\rho^F) \oplus \mathcal{P}_2) \times (\mathcal{P}(K_\rho^F) \oplus \mathcal{P}_1)$ spanned by the pairs

$$\left(\begin{bmatrix} K_\rho^F(\cdot, z) p \\ q \end{bmatrix}, \begin{bmatrix} \rho(z_1, z) K_\rho^F(\cdot, z) - \rho(z_1, z_1) K_\rho^F(\cdot, z_1) \\ \sqrt{\rho(z_1, z_1)} (F(z)^{[*]} - F(z_1)^{[*]}) \end{bmatrix} \frac{p}{\rho(z_1, z)\sigma_\rho^{z_0}(z, z_1)^*} \right.$$
$$\left. + \begin{bmatrix} \sqrt{\rho(z_1, z_1)} K_\rho^F(\cdot, z_1) \\ F(z_1)^{[*]} \end{bmatrix} q \right)$$

as z runs through Ω_1', and p and q through $\mathcal{P}_2$. Note that dom$\mathcal{R}$ is dense in $\mathcal{P}(K_\rho^F) \oplus \mathcal{P}_2$.

Furthermore, for every point $z \in \Omega_1'$ and for every pair $p, q \in \mathcal{P}_2$ it holds that

$$\left\langle \frac{\rho(z_1, z) K_\rho^F(\cdot, z) - \rho(z_1, z_1) K_\rho^F(\cdot, z_1)}{\rho(z_1, z)\sigma_\rho^{z_0}(z, z_1)^*} p, \sqrt{\rho(z_1, z_1)} K_\rho^F(\cdot, z_1) q \right\rangle_{\mathcal{P}(K_\rho^F)}$$
$$+ \left\langle \sqrt{\rho(z_1, z_1)} \frac{F(w)^{[*]} - F(z_1)^{[*]}}{\rho(z_1, z)\sigma_\rho^{z_0}(z, z_1)^*} p, F(z_1)^{[*]} q \right\rangle_{\mathcal{P}_1}$$
$$= \frac{\sqrt{\rho(z_1, z_1)}}{\rho(z_1, z)\sigma_\rho^{z_0}(z, z_1)^*} \langle p, (\rho(z, z_1) K_\rho^F(z, z_1) - \rho(z_1, z_1) K_\rho^F(z_1, z_1)) q \rangle_{\mathcal{P}_2}$$
$$+ \frac{\sqrt{\rho(z_1, z_1)}}{\rho(z_1, z)\sigma_\rho^{z_0}(z, z_1)^*} \langle p, (F(z) F(z_1)^{[*]} - F(z_1) F(z_1)^{[*]}) q \rangle_{\mathcal{P}_2} = 0.$$

Similarly, for $p, q \in \mathcal{P}_2$ and $z, w \in \Omega_1'$ it holds that

$$\begin{aligned}&\Bigg\langle \frac{\rho(z_1,z)K_\rho^F(\cdot,z)-\rho(z_1,z_1)K_\rho^F(\cdot,z_1)}{\rho(z_1,z)\sigma_\rho^{z_0}(z,z_1)^*}p,\\ &\quad\frac{\rho(z_1,w)K_\rho^F(\cdot,w)-\rho(z_1,z_1)K_\rho^F(\cdot,z_1)}{\rho(z_1,w)\sigma_\rho^{z_0}(w,z_1)^*}q\Bigg\rangle_{\mathcal{P}(K_\rho^F)}\\ &\quad+\left\langle\sqrt{\rho(z_1,z_1)}\frac{F(z)^{[*]}-F(z_1)^{[*]}}{\rho(z_1,z)\sigma_\rho^{z_0,z_1}(z)^*}p,\sqrt{\rho(z_1,z_1)}\frac{F(w)^{[*]}-F(z_1)^{[*]}}{\rho(z_1,w)\sigma_\rho^{z_0}(w,z_1)^*}q\right\rangle_{\mathcal{P}_1}\\ &=\frac{\rho(z_1,z)\rho(w,z_1)-\rho(z_1,z_1)\rho(w,z)}{\rho(z_1,z)\rho(w,z_1)\sigma_\rho^{z_0}(z,z_1)^*\sigma_\rho^{z_0}(w,z_1)}\langle p, K_\rho^F(z,w)q\rangle_{\mathcal{P}_2}.\end{aligned}$$

In view of Lemma 4.8, this last expression is equal to

$$\langle p, K_\rho^F(z,w)q\rangle_{\mathcal{P}_2}=\langle K_\rho^F(\cdot,z)p, K_\rho^F(\cdot,w)q\rangle_{\mathcal{P}(K_\rho^F)}.$$

Finally, for $p, q \in \mathcal{P}_2$ we have

$$\begin{aligned}&\left\langle\sqrt{\rho(z_1,z_1)}K_\rho^F(\cdot,z_1)p,\sqrt{\rho(z_1,z_1)}K_\rho^F(\cdot,z_1)q\right\rangle_{\mathcal{P}(K_\rho^F)}+\langle F(z_1)^{[*]}p, F(z_1)^{[*]}q\rangle_{\mathcal{P}_1}\\ &\quad=\langle p,(\rho(z_1,z_1)K_\rho^F(z_1,z_1)+F(z_1)F(z_1)^{[*]})q\rangle_{\mathcal{P}_2}=\langle p,q\rangle_{\mathcal{P}_2}.\end{aligned}$$

Thus the linear relation $\mathcal{R}$ is isometric. Since $\mathcal{P}(K_\rho^F)\oplus\mathcal{P}_1$ and $\mathcal{P}(K_\rho^F)\oplus\mathcal{P}_2$ are Pontryagin spaces with the same index, it follows from a theorem of Shmul'yan (see [11, Theorem 1.4.1], [40]) that the closure of the linear relation $\mathcal{R}$ is the graph of an isometry from $\mathcal{P}(K_\rho^F)\oplus\mathcal{P}_2$ to $\mathcal{P}(K_\rho^F)\oplus\mathcal{P}_1$. The adjoint operator is precisely the coisometric operator

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{P}(K_\rho^F)\\ \mathcal{P}_1\end{bmatrix}\longrightarrow\begin{bmatrix}\mathcal{P}(K_\rho^F)\\ \mathcal{P}_2\end{bmatrix},$$

given by (4.9)–(4.12).

Indeed, for every point $z \in \Omega_1'$ and every vector $p \in \mathcal{P}_2$ it holds that

$$\begin{aligned}\langle p,(Af)(z)\rangle_{\mathcal{P}_2}&=\left\langle p,\frac{\rho(z,z_1)f(z)-\rho(z_1,z_1)f(z_1)}{\rho(z,z_1)\sigma_\rho^{z_0}(z,z_1)}\right\rangle_{\mathcal{P}_2}\\ &=\left\langle\frac{\rho(z_1,z)K_\rho^F(\cdot,z)-\rho(z_1,z_1)K_\rho^F(\cdot,z_1)}{\rho(z_1,z)\sigma_\rho^{z_0}(z,z_1)^*}p,f\right\rangle_{\mathcal{P}(K_\rho^F)}.\end{aligned}$$

The rest of the formulas (4.9)–(4.12) is verified similarly. □

Proof of Theorem 4.10. If $F(z)$ and $S(z)$ are two $\mathbf{L}(\mathcal{P}_1,\mathcal{P}_2)$-valued functions, which are analytic in Ω_1 and Ω_2, respectively, and satisfy the relation (4.8), then, as follows from Lemma 4.8, the kernels $K_\rho^F(z,w)$ and $K_{\rho_{\mathbf{S}}}^S(z,w)$ are related by

$$K_\rho^F(z,w)=\frac{\sqrt{\rho(z_1,z_1)}}{\rho(z,z_1)}K_{\rho_S}^S\left(\sigma_\rho^{z_0}(z,z_1),\sigma_\rho^{z_0}(w,z_1)\right)\frac{\sqrt{\rho(z_1,z_1)}}{\rho(z_1,w)}. \tag{4.13}$$

In particular, the kernels $K_\rho^F(z,w)$ and $K_{\rho_{\mathbf{S}}}^S(z,w)$ have the same number of negative squares in Ω' and Ω'', respectively.

Thus it suffices to show that if $F(z)$ is analytic in Ω', and the kernel $K_\rho^F(z,w)$ has κ negative squares there, then $F(z)$ can be represented in the form (4.8), where $S(z)$ is analytic in Ω''.

Since the point z_1 in Lemma 4.11 is arbitrary in Ω', it follows from (4.9) that for every $w \in \Omega'$ the formula

$$(R_w f)(z) := \frac{\rho(z,w)f(z) - \rho(w,w)f(w)}{\rho(z,w)\sigma_\rho^{z_0}(z,w)}$$

defines a continuous operator R_w on $\mathcal{P}(K_\rho^F)$. The operator R_w plays the role of the backward-shift operator in the present setting (see [13]). For every pair of points $z, w \in \Omega'$ it holds that

$$\left(I - \sigma_\rho^{z_0}(z,w)R_w\right)\left(I - \sigma_\rho^{z_0}(w,z)R_z\right) = (1 - |\sigma_\rho^{z_0}(z,w)|^2)I, \tag{4.14}$$

where $I := I_{\mathcal{P}(K_\rho^F)}$. In particular, the operator $I - zR_{z_1}$ is invertible for every $z \in \Omega''$.

Consider now the $L(\mathcal{P}_1, \mathcal{P}_2)$-valued function $S(z)$ defined by

$$S(z) := D + zC(I - zA)^{-1}B, \quad z \in \rho(A),$$

with the operators A, B, C, D from Lemma 4.11. The function $S(z)$ is analytic in Ω'' and, in view of (4.14), for every $f \in \mathcal{P}(K_\rho^F)$ and $z, w \in \Omega'$ it holds that

$$\begin{aligned}\sigma_\rho^{z_0}(z,z_1)C\left(I - \sigma_\rho^{z_0}(z,z_1)A\right)^{-1}B &= \frac{\sigma_\rho^{z_0}(z,z_1)\rho(z_1,z_1)}{1 - |\sigma_\rho^{z_0}(z,z_1)|^2}\frac{\rho(z,z)}{\rho(z_1,z)}\frac{F(z) - F(z_1)}{\rho(z,z_1)\sigma_\rho^{z_0}(z,z_1)}\\ &= F(z) - F(z_1)\\ &= F(z) - D.\end{aligned}$$

Thus

$$F(z) = D + \sigma_\rho^{z_0}(z,z_1)C\left(I - \sigma_\rho^{z_0}(z,z_1)A\right)^{-1}B = S(\sigma_\rho^{z_0}(z,z_1)). \quad \square$$

Remark 4.12. Let $z_1 \in \mathbb{D}$, and let $z_0 \in \mathbb{T}$. If $S(z)$ is an operator-valued function from the class $\mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2)$, and $F(z)$ is defined by (4.8), where $\sigma_\rho^{z_0}(z,w) = \sigma_{\mathbf{S}}^{z_0}(z,w)$, then the relation (4.13) implies that the function $F(z)$ belongs to the class $\mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2)$ as well. Thus

$$\mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2) = \left\{S(\sigma_{\mathbf{S}}^{z_0}(z,z_1)) \,:\, S(z) \in \mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2)\right\}.$$

Remark 4.13. Let Ω' be an open subset of $\Omega_+(\rho)$. In view of Example 4.4, Theorem 4.10 implies that every element $F(z) \in \mathbf{O}_\kappa^{\Omega'}(\rho, \mathcal{P}_1, \mathcal{P}_2)$ admits a unique meromorphic extension in $\Omega_+(\rho)$ with the property that the associated kernel $K_\rho^F(z,w)$ has κ negative squares in any non-empty open subset of $\Omega_+(\rho)$.

Definition 4.14. An $\mathbf{L}(\mathcal{P}_1, \mathcal{P}_2)$-valued function $F(z)$ is said to belong to the class $\mathbf{O}_\kappa(\rho, \mathcal{P}_1, \mathcal{P}_2)$ if it is meromorphic in $\Omega_+(\rho)$ and the associated kernel

$$K_\rho^F(z,w) = \frac{I_{\mathcal{P}_2} - F(z)F(w)^{[*]}}{\rho(z,w)}$$

has κ negative squares in $\Omega_+(\rho)$.

Note that, in view of Remark 4.13,

$$\mathbf{O}_\kappa(\rho, \mathcal{P}_1, \mathcal{P}_2) = \bigcup_{\Omega'} \mathbf{O}_\kappa^{\Omega'}(\rho, \mathcal{P}_1, \mathcal{P}_2),$$

where the union is taken over all open subsets Ω' of $\Omega_+(\rho)$.

Example 4.15. Definition 4.14 implies that if $\mathcal{P}_1 = \mathcal{P}_2 = \mathbb{C}$, then

$$\mathbf{O}_\kappa(\rho, \mathbb{C}, \mathbb{C}) = \Sigma_\kappa(I_2, \rho).$$

Example 4.16. Denote by $\mathbb{C}^2_J$ the vector space $\mathbb{C}^2$ equipped with the indefinite inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{C}^2_J} := \mathbf{v}^* J \mathbf{u}.$$

A $\mathbb{C}^{2\times 2}$-valued function $\Theta(z)$ in $\Omega_+(\rho)$ belongs to the class $\mathbf{O}_\kappa(\rho, \mathbb{C}^2_J, \mathbb{C}^2_J)$ if and only if it is meromorphic and the kernel

$$K_\rho^\Theta(z, w) = \frac{I_2 - \Theta(z)\Theta(w)^{[*]}}{\rho(z, w)}$$

has κ negative squares in $\Omega_+(\rho)$. Here

$$\Theta(w)^{[*]} = J\Theta(w)^* J,$$

where $\Theta(w)^*$ is the adjoint of the matrix $\Theta(w)$ with respect to the usual inner product in $\mathbb{C}^2$. Therefore, for arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{C}^2$ we have

$$\left\langle K_\rho^\Theta(z, w)\mathbf{u}, \mathbf{v} \right\rangle_{\mathbb{C}^2_J} = \left\langle K_{I_2,\rho}^\Theta(z, w) J\mathbf{u}, J\mathbf{v} \right\rangle_{\mathbb{C}^2},$$

and hence

$$\mathcal{U}(I_2, \rho) = \left\{ \Theta(z) \in \bigcup_{\kappa \in \mathbb{N}_0} \mathbf{O}_\kappa(\rho, \mathbb{C}^2_J, \mathbb{C}^2_J) : \mathcal{P}(K_\rho^\Theta) \text{ is finite-dimensional} \right\}.$$

In particular,

$$\mathcal{U}_{\mathbf{S}} = \left\{ \Theta(z) \in \bigcup_{\kappa \in \mathbb{N}_0} \mathbf{S}_\kappa(\mathbb{C}^2_J, \mathbb{C}^2_J) : \mathcal{P}(K_{\rho_{\mathbf{S}}}^\Theta) \text{ is finite-dimensional} \right\}.$$

In view of Remark 4.12 and Lemma 4.9, Theorem 4.10 yields the following characterization of the classes $\mathbf{O}_\kappa(\rho, \mathcal{P}_1, \mathcal{P}_2)$.

Corollary 4.17. *For $z_1 \in \Omega_+(\rho)$, $z_0 \in \Omega_0(\rho)$ it holds that*

$$\mathbf{O}_\kappa(\rho, \mathcal{P}_1, \mathcal{P}_2) = \{S(\sigma_\rho^{z_0}(z, z_1)) : S(z) \in \mathbf{S}_\kappa(\mathcal{P}_1, \mathcal{P}_2)\}.$$

4.3. The classes $\widetilde{\Sigma}_\kappa(Q, \rho)$ and $\mathcal{U}(Q, \rho)$ in terms of generalized Schur functions

Theorem 4.18. *For $z_1 \in \Omega_+(\rho)$, $z_0 \in \Omega_0(\rho)$ the following statements hold.*

(1) *A function $f(z)$ belongs to $\widetilde{\Sigma}_\kappa(Q, \rho)$ if and only if it is of the form*

$$f(z) = T_{Q(z)}\left(s(\sigma_\rho^{z_0}(z, z_1))\right),$$

where $s(z) \in \mathbf{S}_\kappa$.

(2) *A function $F(z)$ belongs to $\mathcal{U}(Q,\rho)$ if and only if it is of the form*

$$F(z) = Q(z)\Theta\left(\sigma_\rho^{z_0}(z,z_1)\right)Q(z)^{-1},$$

where $\Theta(z) \in \mathcal{U}_{\mathbf{S}}$. In this case $F(z)$ is an elementary factor in $\mathcal{U}(Q,\rho)$ if and only if $\Theta(z)$ is an elementary factor in $\mathcal{U}_{\mathbf{S}}$.

Proof. In view of Example 4.15, statement (1) is an immediate consequence of Proposition 4.1 and Corollary 4.17.

In order to prove statement (2), observe that, in view of Example 4.16 and Remark 4.13,

$$\mathcal{U}(I_2,\rho) = \left\{ F(z) \in \bigcup_{\kappa\in\mathbb{N}_0} \mathbf{O}_\kappa(\rho, \mathbb{C}_J^2, \mathbb{C}_J^2) : \mathcal{P}(K_\rho^F) \text{ is finite dimensional} \right\}.$$

According to Corollary 4.17, a function $F(z)$ belongs to the class $\mathbf{O}_\kappa(\rho, \mathbb{C}_J^2, \mathbb{C}_J^2)$ if and only if it is of the form

$$F(z) = \Theta(\sigma^{z_0}(z,z_1)),$$

where $\Theta(z) \in \mathbf{S}_\kappa\left(\mathbb{C}_J^2, \mathbb{C}_J^2\right)$. But in this case

$$K_\rho^F(z,w) = \frac{\sqrt{\rho(z_1,z_1)}}{\rho(z,z_1)} K_{\rho\mathbf{S}}^\Theta(\sigma_\rho^{z_0}(z,z_1), \sigma_\rho^{z_0}(w,z_1)) \frac{\sqrt{\rho(z_1,z_1)}}{\rho(w,z_1)^*};$$

hence $F(z)$ belongs to the class $\mathcal{U}(I_2,\rho)$ if and only if $\Theta(z)$ belongs to the class $\mathcal{U}_{\mathbf{S}}$. Thus

$$\mathcal{U}(I_2,\rho) = \left\{\Theta(\sigma_\rho^{z_0}(z,z_1)) : \Theta(z) \in \mathcal{U}_{\mathbf{S}}\right\}$$

and, as follows from Proposition 4.2,

$$\mathcal{U}(Q,\rho) = \left\{ Q(z)\Theta(\sigma_\rho^{z_0}(z,z_1))Q(z)^{-1} : \Theta(z) \in \mathcal{U}_{\mathbf{S}} \right\}.$$

Finally, if $F(z) \in \mathcal{U}(Q,\rho)$ and $\Theta(z) \in \mathcal{U}_{\mathbf{S}}$ are such that

$$F(z) = Q(z)\Theta(\sigma_\rho^{z_0}(z,z_1))Q(z)^{-1},$$

then

$$K_{Q,\rho}^F(z,w) = \frac{\sqrt{\rho(z_1,z_1)}}{\rho(z,z_1)} Q(z) K_{\mathbf{S}}^\Theta(\sigma_\rho^{z_0}(z,z_1), c\sigma_\rho^{z_0}(w,z_1)) Q(w)^* \frac{\sqrt{\rho(z_1,z_1)}}{\rho(w,z_1)^*};$$

hence the map

$$\mathbf{f}(z) \mapsto \frac{\sqrt{\rho(z_1,z_1)}}{\rho(z,z_1)} Q(z)\mathbf{f}(\sigma_\rho^{z_0}(z,z_1)) \tag{4.15}$$

is unitary from $\mathcal{P}(K_{\mathbf{S}}^\Theta)$ onto $\mathcal{P}(K_{Q,\rho}^F)$. It follows that a subspace of $\mathcal{P}(K_{Q,\rho}^F)$ is of the form $\mathcal{P}(K_{Q,\rho}^{\tilde F})$ for some $\tilde F(z) \in \mathcal{U}(Q,\rho)$ if and only if it is the image under the map (4.15) of a subspace of $\mathcal{P}(K_{\mathbf{S}}^\Theta)$ of the form $\mathcal{P}(K_{\mathbf{S}}^{\tilde\Theta})$, where $\tilde\Theta(z) \in \mathcal{U}_{\mathbf{S}}$ is such that

$$\tilde F(z) = Q(z)\tilde\Theta\left(\sigma_\rho^{z_0}(z,z_1)\right)Q(z)^{-1}. \quad \square$$

5. The Schur transformation and the basic interpolation problem in the class $\widetilde{\Sigma}_\kappa(Q,\rho)$

5.1. The Schur transformation in $\widetilde{\Sigma}_\kappa(Q,\rho)$

Theorem 4.18 implies that with the mapping $\sigma_\rho^{z_0}(z,z_1)$ from Definition 4.5 we can introduce the Schur transformation in the class $\widetilde{\Sigma}_\kappa(Q,\rho)$, centered at an arbitrary point $z_1 \in \Omega_+(\rho) \cap \mathrm{hol}(Q,Q^{-1})$, via the Schur transformation in the class $\mathbf{S}_\kappa$, centered at the point 0; the latter Schur transformation was defined in Definition 3.9. As we shall explain below the definition of the Schur transform $\hat{f}(z)$ of $f(z) \in \widetilde{\Sigma}_\kappa(Q,\rho)$ takes the following form.

Definition 5.1. Let $z_1 \in \Omega_+(\rho) \cap \mathrm{hol}\,(Q,Q^{-1})$ and $z_0 \in \Omega_0(\rho)$. Consider the function $f(z) \in \widetilde{\Sigma}_\kappa(Q,\rho)$ and let $\mathbf{f}(z)$ be a projective representation of $f(z)$. Set

$$\mathbf{u} := JQ(z_1)^* J\mathbf{f}(z_1)^\times$$

and assume that

$$\mathbf{u}^{\times *} JQ(z)^{-1}\mathbf{f}(z) \not\equiv 0.$$

Then the Schur transform $\hat{f}(z)$ of $f(z)$ centered at $z = z_1$ is given by the linear fractional transformation

$$\hat{f}(z) = T_{\Phi(z)}(f(z)),$$

where $\Phi(z)$ is defined as follows.

(1) If $\mathbf{u}^* J\mathbf{u} \neq 0$, then

$$\Phi(z) := \sigma_\rho^{z_0}(z,z_1) I_2 + \left(1 - \sigma_\rho^{z_0}(z,z_1)\right) \frac{Q(z)\mathbf{u}\mathbf{u}^* JQ(z)^{-1}}{\mathbf{u}^* J\mathbf{u}}.$$

(2) If $\mathbf{u}^* J\mathbf{u} = 0$, then

$$\Phi(z) := \sigma_\rho^{z_0}(z,z_1)^k I_2 - \left(p(\sigma_\rho^{z_0}(z,z_1)) - \sigma_\rho^{z_0}(z,z_1)^{2k} p^\#(\sigma_\rho^{z_0}(z,z_1))\right) \times \frac{Q(z)\mathbf{u}\mathbf{u}^* JQ(z)^{-1}}{\mathbf{u}^* P_2\mathbf{u}},$$

where

$$k := \frac{\mu}{\nu},$$

μ is the order of the zero of the function $\mathbf{u}^* JQ(z)^{-1}\mathbf{f}(z)$ at $z = z_1$, ν is the order of zero of $\sigma_\rho^{z_0}(z,z_1)$ at $z = z_1$ and $p(z)$ is the polynomial determined by $\deg(p) < k$ and

$$\frac{\sigma_\rho^{z_0}(z,z_1)^k \mathbf{u}^* P_2 Q(z)^{-1}\mathbf{f}(z)}{\mathbf{u}^* JQ(z)^{-1}\mathbf{f}(z)} = p(\sigma_\rho^{z_0}(z,z_1)) + \mathrm{O}((z-z_1)^\mu), \quad z \to z_0.$$

We explain this definition using the same notation. According to Theorem 4.18, there exists a function $s(z) \in \mathbf{S}_\kappa$ such that

$$f(z) = T_{Q(z)}(s(\sigma_\rho^{z_0}(z,z_1))).$$

Let $\mathbf{s}(z)$ be a projective representation of $s(z)$. Then there is a meromorphic function $q(z)$ in $\Omega_+(\rho)$ which is analytic and non-zero at z_1 such that

$$Q(z)\mathbf{s}(\sigma_\rho^{z_0}(z,z_1)) = \mathbf{f}(z)q(z)$$

or, equivalently (by (2.1)),

$$\mathbf{s}(\sigma_\rho^{z_0}(z, z_1))^{\times *} = \frac{q(z)}{\det Q(z)} \mathbf{f}(z)^{\times *} J Q(z) J.$$

In particular, the vector $\mathbf{s}(0)^\times = \mathbf{s}(\sigma_\rho^{z_0}(z_1, z_1))^{\times *}$ in Definition 3.9, and the vector $\mathbf{u} = JQ(z_1)^* J\mathbf{f}(z_1)^\times$ in Definition 5.1 are collinear, and therefore the assumption

$$\mathbf{u}^{\times *} J Q(z)^{-1} \mathbf{f}(z) \not\equiv 0$$

in Definition 5.1 is equivalent to the assumption (3.31) in Definition 3.9. Hence one may consider the Schur transform $\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ of the function $s(z)$ centered at $z = 0$. Define

$$\widetilde{f}(z) := T_{Q(z)}(\hat{s}(\sigma_\rho^{z_0}(z, z_1))).$$

Theorem 4.18 implies $\widetilde{f}(z) \in \widetilde{\Sigma}_{\hat{\kappa}}(Q, \rho)$. By Definition 3.9, $\hat{s}(z) = T_{\Phi_0(z)}(s(z))$ with $\Phi_0(z)$ as in (3.32) in case (1) and as in (3.33) in case (2). Hence

$$\widetilde{f}(z) = T_{Q(z)\Phi_0(\sigma_\rho^{z_0}(z,z_1))Q(z)^{-1}},$$

and since $\Phi(z)$ in Definition 5.1 is equal to $Q(z)\Phi_0(\sigma_\rho^{z_0}(z, z_1))Q(z)^{-1}$ we see that $\widetilde{f}(z)$ is the Schur transform $\hat{f}(z)$ of $f(z)$.

It remains to explain the number k in Definition 5.1 in the case when the vector $\mathbf{u}$ is J-neutral. The number μ there is the order of the zero of

$$\mathbf{u}^* J \mathbf{s}(\sigma_\rho^{z_0}(z, z_1)) = \mathbf{u}^* J Q(z)^{-1} \mathbf{f}(z) q(z)$$

at $z = z_1$ and, since ν is the order of the zero of $\sigma_\rho^{z_0}(z, z_1)$ at $z = z_1$, μ is an integer multiple of ν and their quotient $k = \mu/\nu$ is the order of the zero of $\mathbf{u}^* J\mathbf{s}(z)$ at $z = 0$ precisely as in Definition 3.9.

Example 5.2. A special case of Definition 5.1 is the Schur transformation in $\mathbf{S}_\kappa$, centered at an arbitrarily given point $z_1 \in \mathbb{D}$. Set

$$z_0 := \frac{1 + z_1}{1 + z_1^*},$$

so that

$$\sigma_{\mathbf{S}}^{z_0}(z, z_1) = \frac{z - z_1}{1 - z z_1^*}.$$

Consider $s(z) \in \mathbf{S}_\kappa$ with a projective representation $\mathbf{s}(z)$, and set

$$\mathbf{u} := \mathbf{s}(z_1)^\times.$$

Then $\mathbf{u}^{\times *} J\mathbf{s}(z) \equiv 0$ if and only if $s(z)$ is a unimodular constant. If this is not the case then the Schur transform $\hat{s}(z)$ of $s(z)$ at $z = z_1$ is defined and given by the linear fractional transformation

$$\hat{s}(z) = T_{\Phi(z)}(s(z)),$$

where $\Phi(z)$ is defined as follows.

(1) If $\mathbf{u}^* J\mathbf{u} \neq 0$, then

$$\Phi(z) = \sigma_{\mathbf{S}}^{z_0}(z, z_1) I_2 + \left(1 - \sigma_{\mathbf{S}}^{z_0}(z, z_1)\right) \frac{\mathbf{u}\mathbf{u}^* J}{\mathbf{u}^* J\mathbf{u}}.$$

(2) If $\mathbf{u}^* J\mathbf{u} = 0$, then

$$\Phi(z) = \sigma_{\mathbf{S}}^{z_0}(z, z_1)^k I_2 - \left(p(\sigma_{\mathbf{S}}^{z_0}(z, z_1)) - \sigma_{\mathbf{S}}^{z_0}(z, z_1)^{2k} p^{\#}(\sigma_{\mathbf{S}}^{z_0}(z, z_1))\right)\frac{\mathbf{u}\mathbf{u}^* J}{\mathbf{u}^* P_2\mathbf{u}},$$

where k is the order of the zero of $\mathbf{u}^* J\mathbf{s}(z)$ at $z = z_1$ and $p(z)$ is the polynomial determined by $\deg(p) < k$ and

$$\frac{\sigma_{\mathbf{S}}^{z_0}(z, z_1)^k \mathbf{u}^* P_2\mathbf{s}(z)}{\mathbf{u}^* J\mathbf{s}(z)} = p(\sigma_{\mathbf{S}}^{z_0}(z, z_1)) + \mathrm{O}((z - z_1)^k), \quad z \to z_1. \tag{5.1}$$

The expression for $\Phi(z)$ in case (2) can be made more explicit if one observes that (5.1) can be re-written as

$$\frac{\mathbf{u}^* P_2\mathbf{s}(z)(z - z_1)^k}{\mathbf{u}^* J\mathbf{s}(z)(1 - zz_1^*)} = p_1(z) + \mathrm{O}((z - z_1)^k), \quad z \to z_1, \tag{5.2}$$

where

$$p_1(z) := (1 - zz_1^*)^{k-1} p\left(\sigma_{\mathbf{S}}^{z_0}(z, z_1)\right).$$

In particular, $p_1(z)$ is a polynomial of degree $< k$ and is, therefore, uniquely determined by (5.2). Note also that $p_1(z_1) \neq 0$. Next, observe that

$$p^{\#}(\sigma_{\mathbf{S}}(z, z_1)) = \frac{p_1^{\#}(z) z^{k-1}}{(z - z_1)^{k-1}}.$$

Thus $\Phi(z)$ in this case takes the form

$$\Phi(z) = \left(\frac{z - z_1}{1 - zz_1^*}\right)^k I_2 - \frac{p_1(z)(1 - zz_1^*)^{k+1} - z^{k-1} p_1^{\#}(z)(z - z_1)^{k+1}}{(1 - zz_1^*)^{2k}} \frac{\mathbf{u}\mathbf{u}^* J}{\mathbf{u}^* P_2\mathbf{u}}.$$

If $z_1 = 0$, then this $\Phi(z)$ coincides with $\Phi(z)$ in (3.33).

Example 5.3. Definition 5.1 can also be used to introduce the Schur transformation in the class $\widetilde{\mathbf{N}}_\kappa$ centered at an arbitrary point $z_1 \in \mathbb{C}_+$. Let

$$z_0 = \mathrm{Re}\, z_1,$$

so that

$$\sigma_{\mathbf{N}}^{z_0}(z, z_1) = \frac{z_1 - z}{z - z_1^*}.$$

Consider $n(z) \in \widetilde{\mathbf{N}}_\kappa$ and let $\mathbf{n}(z)$ be a projective representation of $n(z)$. Set

$$\mathbf{u} := J Q_{\mathbf{N}}^* J\mathbf{v}, \qquad \mathbf{v} := \mathbf{n}(z_1)^{\times}.$$

Since

$$Q_{\mathbf{N}}^* = 2Q_{\mathbf{N}}^{-1} \quad \text{and} \quad Q_{\mathbf{N}} J Q_{\mathbf{N}}^* = -2J_{\mathbf{N}},$$

the following identities hold:

$$Q_{\mathbf{N}}\mathbf{u} = 2J J_{\mathbf{N}}\mathbf{v}, \qquad \mathbf{u}^{\times *} J Q_{\mathbf{N}}^{-1} = -i\mathbf{v}^{\times *} J_{\mathbf{N}}, \qquad \mathbf{u}^* J\mathbf{u} = 2\mathbf{v}^* J_{\mathbf{N}}\mathbf{v},$$

$$\mathbf{u}^* J Q_{\mathbf{N}}^{-1} = \mathbf{v}^* J, \qquad \mathbf{u}^* P_2 Q_N^{-1} = -\mathbf{v}^* J\frac{I_2 + J_{\mathbf{N}}}{2}, \qquad \mathbf{u}^* P_2\mathbf{u} = \mathbf{v}^*(I_2 - J_{\mathbf{N}})\mathbf{v}.$$

In particular,

$$\mathbf{u}^{\times *} J Q_{\mathbf{N}}^{-1} \mathbf{n}(z) \equiv 0$$

if and only if either $n(z) \equiv \infty$ or $n(z)$ is a real constant. If $n(z)$ is neither ∞, nor a real constant, then its Schur transform $\hat{n}(z)$ at $z = z_1$ is defined and given by the linear fractional transformation

$$\hat{n}(z) = T_{\Phi(z)}(n(z)),$$

where $\Phi(z)$ is defined as follows.

(1) If $\mathbf{v}^* J_{\mathbf{N}} \mathbf{v} \neq 0$, then

$$\Phi(z) = \sigma_{\mathbf{N}}^{z_0}(z, z_1) I_2 + (1 - \sigma_{\mathbf{N}}^{z_0}(z, z_1)) \frac{J J_{\mathbf{N}} \mathbf{v}\mathbf{v}^* J}{\mathbf{v}^* J_{\mathbf{N}} \mathbf{v}}.$$

(2) If $\mathbf{v}^* J_{\mathbf{N}} \mathbf{v} = 0$, then

$$\Phi(z) = \sigma_{\mathbf{N}}^{z_0}(z, z_1)^k I_2 + \left(p(\sigma_{\mathbf{N}}^{z_0}(z, z_1)) - \sigma_{\mathbf{N}}^{z_0}(z, z_1)^{2k} p^{\#}(\sigma_{\mathbf{N}}^{z_0}(z, z_1))\right) \frac{2 J J_{\mathbf{N}} \mathbf{v}\mathbf{v}^* J}{\mathbf{v}^*(I - J_{\mathbf{N}})\mathbf{v}},$$

where k is the order of the zero of $\mathbf{v}^* J \mathbf{n}(z)$ at $z = z_1$ and $p(z)$ is the polynomial determined by $\deg(p) < k$ and

$$\frac{\sigma_{\mathbf{N}}^{z_0}(z, z_1)^k \mathbf{v}^* J (I_2 + J_{\mathbf{N}}) \mathbf{n}(z)}{2 \mathbf{v}^* J \mathbf{n}(z)} = p(\sigma_{\mathbf{N}}^{z_0}(z, z_1)) + \mathrm{O}((z - z_1)^k), \quad z \to z_1. \tag{5.3}$$

The expression for $\Phi(z)$ in case (2) can be made more explicit by re-writing (5.3) as

$$\frac{\mathbf{v}^* J (I + J_{\mathbf{N}}) \mathbf{n}(z)(z - z_1)^k}{2 \mathbf{v}^* J \mathbf{n}(z)(z - z_1^*)} = p_1(z) + \mathrm{O}((z - z_1)^k), \quad z \to z_1, \tag{5.4}$$

where

$$p_1(z) := (-1)^k (z - z_1^*)^{k-1} p(\sigma_{\mathbf{N}}^{z_0}(z, z_1)).$$

In particular, $p_1(z)$ is a polynomial of degree $< k$ and is, therefore, uniquely determined by (5.4). Note that $p_1(z_1) \neq 0$. Since

$$p^{\#}(\sigma_{\mathbf{N}}^{z_0}(z, z_1)) = \frac{(-1)^k p_1(z^*)^*}{(z - z_1)^{k-1}},$$

$\Phi(z)$ in this case takes the form (up to multiplication by the scalar $(-1)^k$):

$$\Phi(z) = \left(\frac{z - z_1}{z - z_1^*}\right)^k I_2 + \frac{p_1(z)(z - z_1^*)^{k+1} - p_1(z^*)^*(z - z_1)^{k+1}}{(z - z_1^*)^{2k}} \frac{2 J J_{\mathbf{N}} \mathbf{v}\mathbf{v}^* J}{\mathbf{v}^*(I - J_{\mathbf{N}})\mathbf{v}}.$$

5.2. The basic interpolation problem in the class $\widetilde{\Sigma}_\kappa(Q, \rho)$

Theorem 5.4. *Let $z_1 \in \Omega_+(\rho) \cap \mathrm{hol}(Q, Q^{-1})$, $z_0 \in \Omega_0(\rho)$, $\mathbf{u}_0 \in \mathbb{C}^2$ and assume $\mathbf{u}_0$ is not J-neutral. Denote by $\Theta(z)$ the $\mathbb{C}^{2\times 2}$-valued function*

$$\Theta(z) = I_2 - (1 - \sigma_{\rho}^{z_0}(z, z_1)) \frac{Q(z) \mathbf{u}_0 \mathbf{u}_0^* J Q(z)^{-1}}{\mathbf{u}_0^* J \mathbf{u}_0}. \tag{5.5}$$

Then the following statements for a complex function $f(z)$ are equivalent.

(1) *$f(z) \in \widetilde{\Sigma}_\kappa(Q,\rho)$ and any projective representation $\mathbf{f}(z)$ of $f(z)$ satisfies*

$$\mathbf{u}_0^* J Q(z_1)^{-1}\mathbf{f}(z_1) = 0.$$

(2) *$f(z)$ is of the form*

$$f(z) = T_{\Theta(z)}(\hat{f}(z)),$$

where $\hat{f}(z) \in \widetilde{\Sigma}_{\hat{\kappa}}(Q,\rho)$ with

$$\hat{\kappa} := \begin{cases} \kappa, & \mathbf{u}_0^* J\mathbf{u}_0 > 0, \\ \kappa - 1, & \mathbf{u}_0^* J\mathbf{u}_0 < 0, \end{cases} \tag{5.6}$$

and any projective representation $\hat{\mathbf{f}}(z)$ of $\hat{f}(z)$ satisfies

$$\mathbf{u}_0^{\times *} J Q(z_1)^{-1}\hat{\mathbf{f}}(z_1) \neq 0. \tag{5.7}$$

Proof. By Theorem 4.18, statement (1) holds if and only if $f(z)$ is of the form

$$f(z) = T_{Q(z)}(s(\sigma_\rho^{z_0}(z,z_1))),$$

where $s(z) \in \mathbf{S}_\kappa$ and any projective representation $\mathbf{s}(z)$ of $s(z)$ satisfies

$$\mathbf{u}_0^* J\mathbf{s}(0) = 0.$$

According to Theorem 3.4, this is equivalent to $s(z)$ being of the form

$$s(z) = T_{\Theta_0(z)}(\hat{s}(z)),$$

where

$$\Theta_0(z) = I_2 - (1-z)\frac{\mathbf{u}_0\mathbf{u}_0^* J}{\mathbf{u}_0^* J\mathbf{u}_0},$$

$\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ with $\hat{\kappa}$ as in (5.6) and any projective representation $\hat{\mathbf{s}}(z)$ of $\hat{s}(z)$ satisfies

$$\mathbf{u}_0^* J\hat{\mathbf{s}}(0)^\times \neq 0.$$

This in turn holds if and only if the function

$$\hat{f}(z) := T_{Q(z)}(\hat{s}(\sigma_\rho^{z_0}(z,z_1)))$$

belongs to $\widetilde{\Sigma}_{\hat{\kappa}}(Q,\rho)$ and any projective representation $\hat{\mathbf{f}}(z)$ of $\hat{f}(z)$ satisfies (5.7). Finally, note that

$$\Theta(z) := Q(z)\Theta_0(\sigma_\rho^{z_0}(z,z_1))Q(z)^{-1}$$

is the same as in (5.5). $\square$

Theorem 5.5. *Fix $z_1 \in \Omega_+(\rho) \cap \mathrm{hol}(Q,Q^{-1})$, $z_0 \in \Omega_0(\rho)$ and $\mathbf{u}_0 \in \mathbb{C}^2$, and assume $\mathbf{u}_0$ is non-zero and J-neutral. Let k be a positive integer and let $q(z)$ be a complex polynomial with the properties $q(0) \neq 0$ and $\deg(q) < k$. Denote by ν the order of the zero of $\sigma_\rho^{z_0}(z,z_1)$ at $z = z_1$ and by $\Theta(z)$ the $\mathbb{C}^{2\times 2}$-valued function*

$$\Theta(z) := \sigma_\rho^{z_0}(z,z_1)^k I_2 + (p(\sigma_\rho^{z_0}(z,z_1)) - \sigma_\rho^{z_0}(z,z_1)^{2k} p^\#(\sigma_\rho^{z_0}(z,z_1))) \times \frac{Q(z)\mathbf{u}_0\mathbf{u}_0^* J Q(z)^{-1}}{\mathbf{u}_0^* P_2\mathbf{u}_0}, \tag{5.8}$$

where $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \text{and} \quad p(z)q(z) = 1 + \mathrm{O}(z^k), \quad z \to 0. \tag{5.9}$$

Then for a complex function $f(z)$ the following statements are equivalent.

(1) *$f(z) \in \widetilde{\Sigma}_\kappa(Q, \rho)$ and any projective representation $\mathbf{f}(z)$ of $f(z)$ satisfies*

$$\frac{\mathbf{u}_0^* J Q(z)^{-1}\mathbf{f}(z)}{\mathbf{u}_0^* P_2 Q(z)^{-1}\mathbf{f}(z)} = \sigma_\rho^{z_0}(z, z_1)^k q(\sigma_\rho^{z_0}(z, z_1)) + \mathrm{O}((z - z_1)^{2k\nu}) \quad \text{as } z \to z_1.$$

(2) *$f(z)$ is of the form*

$$s(z) = T_{\Theta(z)}(\hat{f}(z)),$$

where $\hat{f}(z) \in \widetilde{\Sigma}_{\hat\kappa}(Q, \rho)$ with

$$\hat\kappa = \kappa - k, \tag{5.10}$$

and any projective representation $\hat{\mathbf{f}}(z)$ of $\hat{f}(z)$ satisfies

$$\mathbf{u}_0^* J Q(z_1)^{-1}\hat{\mathbf{f}}(z_1) \neq 0. \tag{5.11}$$

Proof. The proof parallels that of Theorem 5.4. First, observe that statement (1) holds if and only if $f(z)$ is of the form

$$f(z) = T_{Q(z)}(s(\sigma_\rho^{z_0}(z, z_1))),$$

where $s(z) \in \mathbf{S}_\kappa$, and any projective representation $\mathbf{s}(z)$ of $s(z)$ satisfies

$$\frac{\mathbf{u}_0^* J \mathbf{s}(z)}{\mathbf{u}_0^* P_2 \mathbf{s}(z)} = z^k q(z) + \mathrm{O}(z^{2k}), \quad z \to 0.$$

According to Theorem 3.6, this is the case if and only if $s(z)$ is of the form

$$s(z) = T_{\Theta_0(z)}(\hat{s}(z)),$$

where

$$\Theta_0(z) := z^k I_2 + (p(z) - z^{2k} p^{\#}(z)) \frac{\mathbf{u}_0 \mathbf{u}_0^* J}{\mathbf{u}_0^* P_2 \mathbf{u}_0},$$

$p(z)$ is the polynomial determined by (5.9), $\hat{s}(z) \in \mathbf{S}_{\hat\kappa}$ with $\hat\kappa$ as in (5.10), and any projective representation $\hat{\mathbf{s}}(z)$ of $\hat{s}(z)$ satisfies the condition

$$\mathbf{u}_0^* J \hat{\mathbf{s}}(0) \neq 0.$$

This last condition holds if and only if the function

$$\hat{f}(z) = T_{Q(z)}(\hat{s}(\sigma_\rho^{z_0}(z, z_1)))$$

belongs to $\widetilde{\Sigma}_{\hat\kappa}(Q, \rho)$ and any projective representation $\hat{\mathbf{f}}(z)$ of $\hat{f}(z)$ satisfies condition (5.11). Finally, note that $\Theta(z)$ in (5.8) and Θ_0 are related by the formula

$$\Theta(z) = Q(z)\Theta_0(\sigma_\rho^{z_0}(z, z_1))Q(z)^{-1}. \quad \square$$

6. The Schur transformation and the basic interpolation problem at a boundary point

6.1. Notation

For $k \in \mathbb{N}$, $z_0 \in \mathbb{T}$ and a polynomial $q(z)$ with $\deg(q) < k$ and expanded around z_0:

$$q(z) = \sum_{j=0}^{k-1} q_j (z - z_0)^j$$

we denote by $\Delta_k(q, z_0)$ the $k \times k$ Toeplitz matrix

$$\Delta_k(q, z_0) := \begin{bmatrix} q_0 & 0 & \cdots & 0 \\ q_1 & q_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ q_{k-1} & \cdots & q_1 & q_0 \end{bmatrix}$$

and define the $k \times k$ matrices $B_k(z_0)$ and E_k by

$$B_k(z_0) = \begin{bmatrix} 0 & \cdots & 0 & \binom{k-1}{0} z_0^{2k-1} \\ \vdots & \iddots & \iddots & \binom{k-1}{1} z_0^{2k-2} \\ 0 & \binom{1}{0} z_0^3 & \iddots & \vdots \\ \binom{0}{0} z_0 & \binom{1}{1} z_0^2 & \cdots & \binom{k-1}{k-1} z_0^k \end{bmatrix},$$

$$E_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (-1)^{k-1} \end{bmatrix}.$$

That the product $\Delta_k(q, z_0) B_k(z_0) E_k$ is Hermitian is an important condition in solving boundary interpolation problems for generalized Schur functions; see [9]. This condition can be characterized in terms of the following relation between the polynomial $q(z)$ and the polynomial

$$(z z_0)^{k-1} q^{\#}(z) = (z z_0)^{k-1} q(1/z^*)^* = \sum_{j=0}^{k-1} q_j^* (z z_0)^{k-j-1} (z_0 - z)^j.$$

Lemma 6.1. *The matrix $\Delta_k(q, z_0) B_k(z_0) E_k$ is Hermitian if and only if*

$$(z z_0)^{k-1} q^{\#}(z) = (-1)^{k-1} (z z_0)^{2k-1} q(z) + \mathrm{O}((z - z_0)^k), \quad z \to z_0.$$

Proof. If we denote by $\mathbf{v}_k(z)$ the $1 \times k$ vector $\mathbf{v}_k(z) = \begin{bmatrix} 1 & z & \cdots & z^{k-1} \end{bmatrix}$ and by $\mathbf{O}((z - z_0)^t)$, $t \in \mathbb{N}$, any $1 \times k$ vector whose entries are $\mathrm{O}((z - z_0)^t)$ as $z \to z_0$, then

$$\mathbf{v}_k(z - z_0) \Delta_k(q, z_0) = q(z) \mathbf{v}_k(z - z_0) + \mathbf{O}((z - z_0)^k), \quad z \to z_0,$$

$$\mathbf{v}_k(z - z_0) B_k(z_0) = z_0 (z - z_0)^{k-1} \mathbf{v}_k\left(\frac{z z_0}{z - z_0}\right),$$

$$\mathbf{v}_k\left(\frac{z z_0}{z - z_0}\right) E_k = \mathbf{v}_k\left(\frac{z z_0}{z_0 - z}\right).$$

For the second equality we used the binomial theorem. Combining these relations we find that as $z \to z_0$

$$\mathbf{v}_k(z-z_0)\Delta_k(q,z_0)B_k(z_0)E_k = z_0(z-z_0)^{k-1}q(z)\mathbf{v}_k\left(\frac{zz_0}{z_0-z}\right) + \mathbf{O}((z-z_0)^k). \quad (6.1)$$

Similarly, as $z \to z_0$

$$\begin{aligned}
\mathbf{v}_k(z-z_0)E_k &= \mathbf{v}_k(z_0-z),\\
\mathbf{v}_k(z_0-z)B_k(z_0)^* &= \frac{1}{z}\left(\frac{z_0-z}{zz_0}\right)^{k-1}\mathbf{v}_k\left(\frac{zz_0}{z_0-z}\right) + \mathbf{O}((z-z_0)^k),\\
\mathbf{v}_k\left(\frac{zz_0}{z_0-z}\right)\Delta_k(q,z_0)^* &= q^{\#}(z)\mathbf{v}_k\left(\frac{zz_0}{z_0-z}\right) + \mathbf{O}(z-z_0).
\end{aligned}$$

For the second equality we used $|z_0| = 1$ and that for $s, t \in \mathbb{N}_0$

$$\frac{1}{(1-z)^{s+1}} = \sum_{\ell=0}^{\infty}\binom{\ell+s}{s}z^\ell = \sum_{\ell=0}^{t}\binom{\ell+s}{s}z^\ell + \mathrm{O}(z^{t+1}), \quad z \to 0.$$

Combining the above three relations we see that as $z \to z_0$

$$\begin{aligned}
\mathbf{v}_k(z-z_0)E_kB_k(z_0)^*\Delta_k(q,z_0)^* = {} & \frac{1}{z}\left(\frac{z_0-z}{zz_0}\right)^{k-1}q^{\#}(z)\mathbf{v}_k\left(\frac{zz_0}{z_0-z}\right)\\
& + \mathbf{O}((z-z_0)^k). \quad (6.2)
\end{aligned}$$

The lemma follows by equating the equalities (6.1) and (6.2). □

In the next subsections we discuss the Schur transformation centered at a boundary point. For this we introduce some additional notation. Let $z_1 \in \Omega_+(\rho)$ and denote by $\Omega_0'(\rho)$ the following subset of $\Omega_0(\rho)$:

$$\Omega_0'(\rho) := \left\{ z \in \Omega_0(\rho) : \left.\frac{d\sigma_\rho^z(w,z_1)}{dw}\right|_{w=z} \neq 0 \right\}.$$

Note that, in view of Lemma 4.9, $\Omega_0'(\rho)$ does not depend on the choice of z_1.

6.2. *The Schur transformation in $\mathbf{S}_\kappa$ at a boundary point*

We recall from the end of [9, Section 3] and the survey paper [6, Subsection 6.1] the following definition of the Schur transformation in the class $\mathbf{S}_\kappa$ centered at the boundary point $z_0 = 1$ of $\mathbb{D}$.

Definition 6.2. Let $s(z) \in \mathbf{S}_\kappa$ with a projective representation $\mathbf{s}(z)$. Assume that there exist a non-zero J-neutral vector $\mathbf{u} \in \mathbb{C}^2$, a positive integer k and a polynomial $q(z)$ such that

(1) $q(1) \neq 0, \deg(q) < k$,
(2) the $k \times k$ matrix $\Delta_k(q,1)B_k(1)E_k$ is Hermitian and
(3) $\frac{\mathbf{u}^* J\mathbf{s}(z)}{\mathbf{u}^* P_2\mathbf{s}(z)} = (z-1)^k q(z) + \mathrm{O}((z-1)^{2k}),\ z \hat{\to} 1.$

Then the Schur transform $\hat{s}(z)$ of $s(z)$ at $z = 1$ is given by

$$\hat{s}(z) = T_{\Phi(z)}(s(z)),$$

where

$$\Phi(z) = I_2 - \frac{(z+1)p(z)}{(z-1)^k}\frac{\mathbf{u}\mathbf{u}^* J}{\mathbf{u}^* P_2 \mathbf{u}}$$

and $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \text{and} \quad p(z)q(z) = \frac{1}{1+z} + \mathrm{O}((z-1)^k), \quad z \to 1.$$

6.3. The basic interpolation problem in $\mathbf{S}_\kappa$ at a boundary point

The inverse Schur transformation in the class $\mathbf{S}_\kappa$ centered at 1 can be used to describe the solutions of the basic interpolation problem in $\mathbf{S}_\kappa$, which takes in this setting the following form (see [9, Theorem 3.2] and [6, Theorem 6.3]).

Theorem 6.3. *Let $\mathbf{u}_0$ be a non-zero J-neutral vector in $\mathbb{C}^2$, k a positive integer and $q(z)$ a polynomial such that $q(1) \neq 0$, $\deg(q) < k$ and the $k \times k$ matrix $\Delta_k(q,1)B_k(1)E_k$ is Hermitian. Denote by $\Theta(z)$ the $\mathbb{C}^{2\times 2}$-valued function*

$$\Theta(z) := I_2 + \frac{(z+1)p(z)}{(z-1)^k}\frac{\mathbf{u}_0\mathbf{u}_0^* J}{\mathbf{u}_0^* P_2 \mathbf{u}_0},$$

where $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \textit{and} \quad p(z)q(z) = \frac{1}{1+z} + \mathrm{O}((z-1)^k), \quad z \to 1.$$

Then for a function $s(z)$ the following statements are equivalent.

(1) *$s(z) \in \mathbf{S}_\kappa$ and any projective representation $\mathbf{s}(z)$ of $s(z)$ satisfies*

$$\frac{\mathbf{u}_0^* J\mathbf{s}(z)}{\mathbf{u}_0^* P_2\mathbf{s}(z)} = (z-1)^k q(z) + \mathrm{O}((z-1)^{2k}), \quad z \hat{\to} 1.$$

(2) *$s(z)$ is of the form*

$$s(z) = T_{\Theta(z)}(\hat{s}(z)),$$

where $\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ with

$$\hat{\kappa} := \begin{cases} \kappa - \dfrac{k-1}{2}, & k \textit{ odd}, (-1)^{(k-1)/2}q(1) > 0,\\ \kappa - \dfrac{k+1}{2}, & k \textit{ odd}, (-1)^{(k-1)/2}q(1) < 0,\\ \kappa - \dfrac{k}{2}, & k \textit{ even}, \end{cases}$$

and any projective representation $\hat{\mathbf{s}}(z)$ of $\hat{s}(z)$ satisfies

$$\liminf_{z \hat{\to} 1}\left|\frac{\mathbf{u}_0^* J\hat{\mathbf{s}}(z)}{\mathbf{u}_0^* P_2\hat{\mathbf{s}}(z)}\right| > 0.$$

6.4. The Schur transformation in $\widetilde{\Sigma}_\kappa(Q,\rho)$ at a boundary point

The following definition of the Schur transform of $f(z) \in \widetilde{\Sigma}_\kappa(Q,\rho)$ at a boundary point $z_0 \in \Omega_0'(\rho) \cap \mathrm{hol}(Q, Q^{-1})$ is motivated by and obtained from Definition 6.2 in the same way as Definition 5.1 is motivated by and obtained from Definition 3.9. Details can be found in the proof of Theorem 6.5 below.

Definition 6.4. Let $z_1 \in \Omega_+(\rho)$ and $z_0 \in \Omega_0'(\rho) \cap \mathrm{hol}(Q, Q^{-1})$. Consider $f(z) \in \widetilde{\Sigma}_\kappa(Q, \rho)$ and let $\mathbf{f}(z)$ be a projective representation of $f(z)$. Assume that there exist a non-zero J-neutral vector $\mathbf{u} \in \mathbb{C}^2$, a positive integer k and a polynomial $q(z)$ such that

(1) $q(1) \neq 0, \deg(q) < k$,
(2) the $k \times k$ matrix $\Delta_k(q, 1)B_k(1)E_k$ is Hermitian and
(3) $\frac{\mathbf{u}^* J Q(z)^{-1}\mathbf{f}(z)}{\mathbf{u}^* P_2 Q(z)^{-1}\mathbf{f}(z)} = \left(\sigma_\rho^{z_0}(z, z_1) - 1\right)^k q\left(\sigma_\rho^{z_0}(z, z_1)\right) + \mathrm{O}((z - z_0)^{2k}),\ z \hat{\rightarrow} z_0.$

Then the Schur transform $\hat{f}(z)$ of $f(z)$ at $z = z_0$ is given by

$$\hat{f}(z) = T_{\Phi(z)}(f(z)),$$

where

$$\Phi(z) := I_2 - \frac{1 + \sigma_\rho^{z_0}(z, z_1)}{(\sigma_\rho^{z_0}(z, z_1) - 1)^k} p(\sigma_\rho^{z_0}(z, z_1)) \frac{Q(z)\mathbf{u}\mathbf{u}^* J Q(z)^{-1}}{\mathbf{u}^* P_2 \mathbf{u}}$$

and $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \text{and} \quad p(z)\,q(z) = \frac{1}{1 + z} + \mathrm{O}((z - 1)^k), \quad z \rightarrow 1.$$

6.5. *The basic interpolation problem in $\widetilde{\Sigma}_\kappa(Q, \rho)$ at a boundary point*

The following result follows from Theorem 6.3.

Theorem 6.5. *Let $z_1 \in \Omega_+(\rho)$ and $z_0 \in \Omega_0'(\rho) \cap \mathrm{hol}(Q, Q^{-1})$. Let $\mathbf{u}_0 \in \mathbb{C}^2$ be a non-zero J-neutral vector, k a positive integer and $q(z)$ a polynomial such that $q(1) \neq 0, \deg(q) < k$ and the $k \times k$ matrix $\Delta_k(q, 1)B_k(1)E_k$ is Hermitian. Denote by $\Theta(z)$ the $\mathbb{C}^{2\times 2}$-valued function*

$$\Theta(z) := I_2 + \frac{1 + \sigma_\rho^{z_0}(z, z_1)}{(\sigma_\rho^{z_0}(z, z_1) - 1)^k} p(\sigma_\rho^{z_0}(z, z_1)) \frac{Q(z)\mathbf{u}_0\mathbf{u}_0^* J Q(z)^{-1}}{\mathbf{u}_0^* P_2 \mathbf{u}_0}, \tag{6.3}$$

where $p(z)$ is the polynomial determined by

$$\deg(p) < k \quad \textit{and} \quad p(z)q(z) = \frac{1}{1 + z} + \mathrm{O}((z - 1)^k), \quad z \rightarrow 1. \tag{6.4}$$

Then for a function $f(z)$ the following statements are equivalent.

(1) *$f(z) \in \widetilde{\Sigma}_\kappa(Q, \rho)$ and any projective representation $\mathbf{f}(z)$ of $f(z)$ satisfies*

$$\frac{\mathbf{u}_0^* J Q(z)^{-1}\mathbf{f}(z)}{\mathbf{u}_0^* P_2 Q(z)^{-1}\mathbf{f}(z)} = \left(\sigma_\rho^{z_0}(z, z_1) - 1\right)^k q\left(\sigma_\rho^{z_0}(z, z_1)\right) + \mathrm{O}((z - z_0)^{2k}), \quad z \hat{\rightarrow} z_0.$$

(2) *$f(z)$ is of the form*

$$f(z) = T_{\Theta(z)}(\hat{f}(z)),$$

where $\hat{f}(z) \in \widetilde{\Sigma}_{\hat{\kappa}}(Q, \rho)$ *with*

$$\hat{\kappa} := \begin{cases} \kappa - \dfrac{k-1}{2}, & k \text{ odd}, (-1)^{(k-1)/2} q(1) > 0, \\ \kappa - \dfrac{k+1}{2}, & k \text{ odd}, (-1)^{(k-1)/2} q(1) < 0, \\ \kappa - \dfrac{k}{2}, & k \text{ even}, \end{cases} \tag{6.5}$$

and any projective representation $\hat{\mathbf{f}}(z)$ *of* $\hat{f}(z)$ *satisfies*

$$\liminf_{z \hat{\to} z_1} \left| \frac{\mathbf{u}_0^* J Q(z)^{-1} \hat{\mathbf{f}}(z)}{\mathbf{u}_0^* P_2 Q(z)^{-1} \hat{\mathbf{f}}(z)} \right| > 0. \tag{6.6}$$

Proof. The proof parallels that of Theorem 5.5. In view of Theorem 4.18, statement (1) holds if and only if $f(z)$ is a function of the form

$$f(z) = T_{Q(z)}(s(\sigma_\rho^{z_0}(z, z_1))),$$

where $s(z)$ is a function from the class $\mathbf{S}_\kappa$, and any projective representation $\mathbf{s}(z)$ satisfies

$$\frac{\mathbf{u}_0^* J \mathbf{s}(z)}{\mathbf{u}_0^* P_2 \mathbf{s}(z)} = (z-1)^k q(z) + \mathrm{O}((z-1)^{2k}), \quad z \hat{\to} 1.$$

According to Theorem 6.3, this is the case if and only if $s(z)$ is of the form

$$s(z) = T_{\Theta_0(z)}(\hat{s}(z)),$$

where

$$\Theta_0(z) := I_2 + \frac{(1+z)p(z)}{(z-1)^k} \frac{\mathbf{u}_0 \mathbf{u}_0^* J}{\mathbf{u}_0^* P_2 \mathbf{u}_0},$$

$p(z)$ is the polynomial determined by (6.4), $\hat{s}(z) \in \mathbf{S}_{\hat{\kappa}}$ with $\hat{\kappa}$ as in (6.5), and any projective representation $\hat{\mathbf{s}}(z)$ of $\hat{s}(z)$ satisfies

$$\liminf_{z \hat{\to} 1} \left| \frac{\mathbf{u}_0^* J \hat{\mathbf{s}}(z)}{\mathbf{u}_0^* P_2 \hat{\mathbf{s}}(z)} \right| > 0.$$

This last condition holds if and only if the function

$$\hat{f}(z) := T_{Q(z)}(\hat{s}(\sigma_\rho^{z_0}(z, z_1)))$$

belongs to $\widetilde{\Sigma}_{\hat{\kappa}}(Q, \rho)$ and any projective representation $\hat{\mathbf{f}}(z)$ of $f(z)$ satisfies (6.6). Finally, note that the $\mathbb{C}^{2\times 2}$ valued function

$$\Theta(z) := Q(z)\Theta_0(\sigma_\rho^{z_0}(z, z_1))Q(z)^{-1}$$

is the same as in (6.3). □

6.6. An example: generalized Nevanlinna functions

In [8] the following basic interpolation problem is solved: given $k \in \mathbb{N}$ and $k+2$ real numbers $z_0, \nu_0, \nu_k, \ldots, \nu_{2k-1}$ with $\nu_k \neq 0$. Find all generalized Nevanlinna functions $n(z)$ such that

$$n(z) = \nu_0 + \nu_k(z-z_0)^k + \cdots + \nu_{2k-1}(z-z_0)^{2k-1} + \mathrm{O}((z-z_0)^{2k}), \quad z \hat{\to} z_0. \tag{6.7}$$

Let $q_0(z)$ be the polynomial

$$q_0(z) = \nu_k + \nu_{k+1}(z - z_0) + \cdots + \nu_{2k-1}(z - z_0)^{k-1}$$

and denote by H_k the Hankel matrix

$$H_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & \nu_k \\ 0 & 0 & \cdots & \nu_k & \nu_{k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \nu_k & \nu_{k+1} & \cdots & \nu_{2k-2} & \nu_{2k-1} \end{pmatrix}.$$

Then the polynomial $p_0(z)$ satisfying

$$\deg(p_0) < k \quad \text{and} \quad p_0(z)q_0(z) = 1 + \mathrm{O}((z - z_0)^k), \quad z \to z_0,$$

is given by

$$p_0(z) = \begin{bmatrix} (z - z_0)^{k-1} & (z - z_0)^{k-2} & \cdots & 1 \end{bmatrix} H_k^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It is shown in [8, Remark 7.3] that the linear fractional transformation

$$n(z) = T_{\Theta_0(z)}(\hat{n}(z)) \quad \text{with } \Theta_0(z) = I_2 + \frac{p_0(z)}{(z - z_0)^k} \begin{bmatrix} \nu_0 & -\nu_0^2 \\ 1 & -\nu_0 \end{bmatrix} \tag{6.8}$$

gives a one-to-one correspondence between all solutions $n(z)$ of the basic interpolation problem and all generalized Nevanlinna functions $\hat{n}(z)$ satisfying

$$\liminf_{z \hat{\to} z_0} \left| \hat{n}(z) - \nu_0 \right| > 0. \tag{6.9}$$

Moreover, $n(z)$ belongs to $\Sigma_\kappa(Q_{\mathbf{N}}, \rho_{\mathbf{N}})$ if and only if $\hat{n}(z)$ belongs to $\Sigma_{\hat{\kappa}}(Q_{\mathbf{N}}, \rho_{\mathbf{N}})$, where, if $\kappa_-(H_k)$ stands for the number of negative eigenvalues of H_k,

$$\hat{\kappa} = \kappa - \kappa_-(H_k). \tag{6.10}$$

This result is a special case of Theorem 6.5 by setting $Q(z) = Q_{\mathbf{N}}$, $\rho(z, w) = \rho_{\mathbf{N}}(z, w)$, $z_1 = z_0 + i$, and

$$\mathbf{u}_0 = \begin{bmatrix} \nu_0 - i \\ \nu_0 + i \end{bmatrix}.$$

Then $\Omega_0'(\rho) \cap \mathrm{hol}(Q, Q^{-1}) = \mathbb{R}$, $\sigma_\rho^{z_0}(z, z_1) = \sigma_{\rho_{\mathbf{N}}}^{z_0}(z, z_0 + i) = \frac{z_0 - z + i}{z - z_0 + i} =: \sigma(z)$ and

$$\sigma^{-1}(z) = \frac{(z_0 - i)z + z_0 + i}{z + 1}.$$

Finally, we choose the polynomial $q(z)$ in Theorem 6.5 such that $\deg(q) < k$ and

$$q(z) = \frac{2i(-i)^k}{\nu_0^2 + 1} \frac{1}{(z + 1)^k} q_0(\sigma^{-1}(z)) + \mathrm{O}((z - 1)^k), \quad z \to 1.$$

Then $q(1) \neq 0$ and, by Lemma 6.1, the matrix $\Delta_k(q,1)B_k(1)E_k$ is Hermitian if and only if

$$q_0(z^*)^* = q_0(z) + \mathrm{O}((z-1)^k), \quad z \to 1,$$

and this holds because $z_0, \nu_k, \nu_{k+1}, \ldots, \nu_{2k-1} \in \mathbb{R}$. Thus the assumptions of Theorem 6.5 are satisfied. After some calculations we find that the symptotic relation in Theorem 6.5(1) with f and $\mathbf{f}$ replaced by n and $\mathbf{n}$ is equivalent to (6.7), that the polynomial $p(z)$ satisfying (6.4) is given by

$$p(z) = \frac{i^{k-1}(\nu_0^2+1)}{2}(z+1)^{k-1}p_0(\sigma^{-1}(z)),$$

and that $\Theta(z)$ in Theorem 6.5(2) is equal to $\Theta_0(z)$ in (6.8). Moreover, the conditions (6.6) with $\hat{f}(z)$ and $\hat{\mathbf{f}}(z)$ replaced by $\hat{n}(z)$ and $\hat{\mathbf{n}}(z)$ and (6.9) are equivalent and the formulas describing $\hat{\kappa}$, (6.5) and (6.10), coincide.

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